Worst Case Output of Uncertain Systems

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Abstract

The numerical solution of the problem treated in this paper is an important step within a couple of recently developed controller-design procedures, dealing with multivariable or uncertain systems subject to hard-bounded control signals. We determine the worst case output amplitude of stable systems, excited with an input signal that is bounded in amplitude and rate. In the case of SISO systems, this problem can be solved via Linear Programming and a numerical algorithm is presented. This is a significant computational simplification compared to the solution used so far. The same framework can be applied for multivariable systems. A consequence of the Linear Programming approach is that uncertain systems, affinely parameterized in the uncertain parameter vector, can be treated via Quadratic Programming.

Key words: Worst Case Output Amplitude, Hard Constraints, Constraint Control, Saturation Avoidance, Rate Constraints, Linear Systems

1 Introduction and Motivation

The maximum possible output amplitude of a linear system, multivariable or uncertain, is calculated assuming that the input is bounded in amplitude and rate. The problem is motivated by its close connection to the design of controllers for systems with saturating or *hard constrained* input signals. Hence, it has high practical importance as almost all real-life control problems are dominated by hard bounds: valves can only be operated between fully open and fully closed, pumps and compressors have a finite throughput capacity and tanks can only hold a certain volume. Exceeding these prescribed bounds causes unexpected behavior of the system – large overshoots, low performance or (in the worst case) instability. A classical example for the detrimental effect of neglecting constraints is the Chernobyl nuclear power plant disaster in 1986.

Consequently, analysis and design of control systems taking care for such constraints is an extremely active area of research. An overview of constraint control is beyond the scope of this paper, we refer to recent textbooks [10,12] and special issues [1,11]. Another direct and somewhat natural approach is to use Model Predictive Control, see, e.g. [4]. In contrast, indirect schemes as Anti Windup [5,13] adjust an already existing controller, which has been designed without direct consideration of constraints on the control signal.

To solve the constraint control problem in a *linear* framework (the so-called saturation avoidance approach), one implicitly has to restrict the amplitude of all external signals, independent from the technique used in particular. Some approaches, however, impose an additional restriction on the rate of the external signals. In many practical situations, this is a very accurate description of those external signals, *possibly* applied to the control system. In the example of the tank from above, not only the liquid-level is bounded (by the tanks height), additionally the liquid cannot change its level arbitrarily fast. A design, directly based on this description will avoid a conservative control system. Design of optimal controllers has been considered [8] as well as uncertain or multivariable systems [7,9]. A common feature of these design procedures is, that they all rely *heavily* on the computation of the maximum amplitude of the control signal, when the external signal is bounded in amplitude and rate. The underlying idea in the controller design is then to adapt the controller in a correct way, when having calculated the maximum control signal exactly, in order to meet the *prescribed* bounds on the control signal. This adaption scheme could be user-interactive, i.e. of a-posteriori character [7,9] or fully automated within an optimization procedure [8]. The core problem, however, in calculating the maximum amplitudes is: given a transfer function (here: from reference signal to control signal) and the bounds on amplitude and rate of the input signal (here: reference signal), then calculate the maximum possible output amplitude (here: maximum control signal) for all admissible inputs. This computational problem will be solved here.

It has already been pointed out in the late 1950s [2], that restricting the rate of external signals is useful for controller design for process applications. The solution was suggested by constructing the worst case input signal. This task was deemed to be time-consuming and therefore restricted to low order systems. The approach enjoyed a revival in the 1980s and although it was mentioned in the overview work on this line of research [3] that the problem is in principle a *linear programming* one, the numerical solution used there [6] was still a "constructive" one – still computationally consuming so that treating multivariable or uncertain systems was out of focus. This paper is intended to fill this void by formulating the SISO problem as an Linear Programming problem which allows then the effective solution of the multivariable case. To treat uncertain SISO systems is then a *quadratic programming* problem, which is the third contribution of this paper.

2 Problem Statement: SISO Case, no uncertainty

Given a stable LTI SISO system, represented by transfer function Π and impulse response π. Denote the input by ξ and the output by λ. As motivated in the introduction, we pose the following:

Definition 1 (Admissible Input) *Let $\Xi, \dot{\Xi} > 0$. A continuous and piecewise differentiable*[1] *signal ξ with $\xi(t) = 0, t \leq 0$, is called* $(\Xi, \dot{\Xi})$-admissible*, short $\xi \in \mathcal{A}(\Xi, \dot{\Xi})$, iff the following conditions hold:*

$$|\xi(t)| \leq \Xi, \quad t \leq 0, \tag{1}$$
$$|\dot{\xi}(t)| \leq \dot{\Xi}, \quad t \leq 0. \tag{2}$$

As motivated above, we are looking for the maximum possible amplitude $\Lambda_m(t)$ of the output λ (up to time t) for all $(\Xi, \dot{\Xi})$-admissible inputs, i.e.

$$\Lambda_m(t) := \sup_{\xi \in \mathcal{A}(\Xi,\dot{\Xi})} \sup_{0<\tau\leq t} |\lambda(\tau)| = \sup_{\xi \in \mathcal{A}(\Xi,\dot{\Xi})} \sup_{0<\tau\leq t} |\pi(\tau) \star \xi(\tau)|, \tag{3}$$

where "$\star$" denotes convolution: $\pi(t) \star \xi(t) := \int_0^t \pi(\tau)\xi(t-\tau)d\tau$. Well-known from linear system theory is, that for systems with the only input constraint (1), the maximum output amplitude is given by $\Xi \int_0^\infty |\pi(\tau)|d\tau$, produced by the so-called bang-bang input $\xi(t-\tau) = \Xi \cdot \mathrm{sign}(\pi(\tau))$. Hence optimization problem (3) is trivial unless the additional constraint (2) is imposed.

It is more convenient to formulate the problem in terms of the time inverted input signal: introduce $\xi_t(\tau) := \xi(t-\tau)$, so that the output is given by $\lambda(t) = \int_0^t \pi(\tau)\xi_t(\tau)d\tau$. The constraints on the input signal then read as:

[1] *this allows a countable number of time stamps, where ξ is not differentiable.*

$$
\begin{aligned}
|\xi_t(\tau)| &\leq \Xi, \quad \tau < t && (4)\\
|\dot{\xi}_t(\tau)| &\leq \dot{\Xi}, \quad \tau < t && (5)\\
\xi_t(\tau) &= 0, \quad \tau \geq t && (6)
\end{aligned}
$$

Obviously, $\Lambda_m(t)$ as defined in (3) is monotone increasing in t. Therefore, the maximum amplitude has to appear for $t \to \infty$, thus $\Lambda_m = \lim_{t\to\infty} \Lambda_m(t)$ is the worst case output amplitude. We therefore pose the following:

Definition 2 (Worst Case Input and Output) *Suppose there exists an* $(\Xi, \dot{\Xi})$*-admissible input* $\xi_{\infty,o} =: \xi_o$ *with maximum output amplitude* Λ_m*. Then* $-\xi_o$ *produces the maximum output amplitude* Λ_m*, too. For one of them, say* ξ_o*, holds*

$$\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau \geq 0, \tag{7}$$

i.e. the absolute value in (3) is obsolete. ξ_o *is called worst case input, The lhs of (7) is called worst case output.*

The optimization problem as stated in Def. 2 now consists of a *linear* objective (7) (in contrast to the one stated in (3), where the absolute value is present) with linear constraints (4-6). Therefore, the solution ξ_o exists and is unique. The remaining (numerical) problem is the infinite time interval. The following section will elaborate conditions that allow a solution using a *finite* linear program.

3 Properties of the Worst Case Input

The following concept is the main vehicle in order to reduce to problem properly to a finite size:

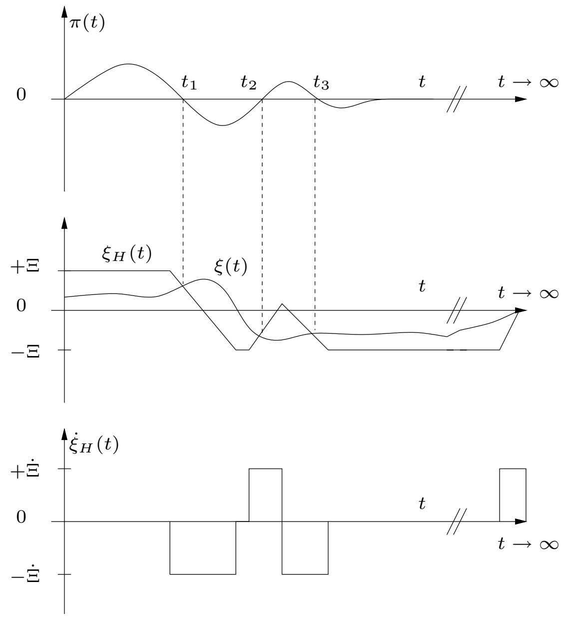

Fig. 1. Construction of the auxiliary input ξ_H for given input ξ.

Algorithm 1 (Construction of auxiliary inputs) *Let $\xi \in \mathcal{A}(\Xi, \dot\Xi)$. Construct an auxiliary input ξ_H for ξ uniquely by the steps given below (see Fig. 1). The set of all auxiliary inputs is denoted by $\mathcal{A}_H(\Xi, \dot\Xi)$.*

1. *Let $\{t_i\}$ the zeros of π. Define $\xi_H(t_i) = \xi(t_i)$.*

2. *If $\pi(t) > 0$ in (t_i, t_{i+1}), let $\dot\xi_H(t) = +\dot\Xi$ in the neighborhood of t_i and $\dot\xi_H(t) = -\dot\Xi$ in the neighborhood of t_{i+1}. In the case that this definition leads to a situation where two "slopes" intersect in some $t_* \in (t_i, t_{i+1})$, let $\dot\xi_H(t) = +\dot\Xi$ in $[t_i, t_*]$ and $\dot\xi_H(t) = -\dot\Xi$ in $[t_*, t_{i+1}]$ respectively. Finally, let $\xi_H = \min\{\xi_H, +\Xi\}$.*

3. *If $\pi(t) < 0$ in (t_i, t_{i+1}), do as in step 2, but with changed signs for $\dot\xi_H$ and resulting obvious modifications.*

4. *Choose $|\dot\xi_H(t)| = \dot\Xi$ for large times t so that $\lim_{t\to\infty} \xi_H(t) = 0$ in order to fulfill (6).*

The following properties of the auxiliary input are clear by construction:

Corollary 1 ***1.*** *Let t_i the zeros of π as in Alg. 1. (1.). Two different inputs $\xi_1, \xi_2 \in \mathcal{A}(\Xi, \dot\Xi)$ with $\xi_1(t_i) = \xi_2(t_i)$ have the same auxiliary input.*

2. *$\xi_H(t) \geq \xi(t)$ for $\pi(t) \geq 0$ and $\xi_H(t) \leq \xi(t)$ for $\pi(t) \leq 0$.*

3. *Fix an admissible input ξ and suppose an arbitrary admissible signal $\xi^* \neq \xi_H$ with property 2, then $\int_0^\infty \pi(\tau)\xi_H(\tau)d\tau > \int_0^\infty \pi(\tau)\xi^*(\tau)d\tau$.*

4. *The maximum width of the pulses of $\dot{\xi}_H$ in Alg. 1 is given by $T = 2 \cdot \Xi/\dot{\Xi}$.*

Corollary 2 *The worst case input is an auxiliary input: $\xi_o \in \mathcal{A}_H(\Xi, \dot{\Xi})$.*

Proof. For all $\xi \in \mathcal{A}(\Xi, \dot{\Xi})$ the following holds by construction of ξ_H, see Cor. 1 (2.):

$$\int_0^\infty \pi(\tau)\xi(\tau)d\tau \leq \int_0^\infty \pi(\tau)\xi_H(\tau)d\tau \tag{8}$$

and "=" holds only for $\xi \equiv \xi_H$ ($\pi \not\equiv 0$). Assume $\xi_o \in \mathcal{A}(\Xi, \dot{\Xi}) \backslash \mathcal{A}_H(\Xi, \dot{\Xi})$, then the construction of an auxiliary input ξ_{oH} is possible (because ξ_o is admissible input). Applying (8) to $\xi = \xi_o$ yields $\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau < \int_0^\infty \pi(\tau)\xi_{oH}(\tau)d\tau$, which contradicts the definition of Λ_m as the maximum output amplitude. Consequently, $\xi_o \in \mathcal{A}_H(\Xi, \dot{\Xi})$. □

We now summarize the necessary properties for the worst case input ξ_o:

Corollary 3 ***1.*** *The derivative of the worst case input has a pulse-shape: $\dot{\xi}_o(t) \in \{\pm\dot{\Xi}, 0\}$, and $\dot{\xi}_o(t) = 0 \Rightarrow |\xi_o(t)| = \Xi$.*

2. *The width of the single pulses of $\dot{\xi}_o$ is constrained by $T = 2 \cdot \Xi/\dot{\Xi}$.*

3. *Two adjacent pulses have different signs.*

4. $\lim_{t\to\infty} \xi_o(t) = 0$ *and* $|\lim_{t\to\infty} \dot{\xi}_o(t)| = \dot{\Xi}$.

Cor. 3 uses properties of $\dot{\xi}_o$. Partial integration in (7) and noting that $\lim_{t\to\infty} \xi_o(t)s(t) = 0$ (Cor. 3. (4.)), gives an expression for the worst case output in this term, where s is the step response of the system (i.e. $\dot{s} = \pi$):

$$\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau = -\int_0^\infty s(\tau)\dot{\xi}_o(\tau)d\tau \tag{9}$$

Under the assumptions that the impulse response π only has a finite number of zeros, say, N, the maximum amplitude Λ_m is given by:

$$\Lambda_m = \dot{\Xi}\sum_{i=1}^{N}(-1)^{i+1+k}\int_{t_i'}^{t_i''} s(t)dt + \Xi(-1)^{N+k}\lim_{t\to\infty} s(t). \tag{10}$$

The last part of the sum exists because the system is stable. The pairs (t_i', t_i'') refer to the unknown positions of the pulses of $\dot{\xi}_o$. Additionally, the sign of the pulses is unknown, therefore $k \in \{0, 1\}$ is added in (10). Obviously the problem is solved, when the exact location of the pulses and their sign are known. Eqn. (10) now enables the reduction to a finite time interval, in the case that the impulse response has an infinite number of zeros. Clearly, the difference between the last two integral terms in (10) becomes arbitrarily small, if $s(\cdot)$ is approximately constant in this interval, because of convergence. As the last part of the sum in (10) is fixed, the problem can be solved applying two linear programs ($k \in \{0, 1\}$ is unknown). This will be outlined in the following section. It is, however, possible to show necessary and sufficient conditions for t_i', t_i'', which may be used to construct the worst case input explicitely:

Remark 1 ([6]) *Necessary condition for $\Lambda_m = \Lambda_m(t_i', t_i'', k)$ in (10) to be a maximum is $s(t_i') = s(t_i'')$, sufficient condition is $k = 0$, if $(t_1, s(t_1))$ is a local maximum, $k = 1$, in the case of a local minimum.*

Remark 2 *For $\dot{\Xi} \to \infty$ (no restriction on the rate), the worst case input boils down to the bang-bang input, as discussed in Sec. 2.*

4 Numerical Solution in the SISO Case

As observed in the previous section, we need to compute the (time inverted) worst case input up to a certain time stamp. Suppose a grid of the non-negative time axis, denoted as $\{t_k\}$ and evaluate the impulse response of the system Π at those time instances, denoted as $\{\pi_k\}$. Then the discrete time version of (7) is

$$\Lambda_m = \sum_{k=0}^{\infty} \pi_k \xi_{o,k}, \tag{11}$$

where $\xi_{o,k}$ is the input sequence, reversed in time. The task is to maximize (11) under the constraints (1,2), which can be approximated for the discrete time case by

$$-\Xi \leq \quad \xi_{o,k} \quad \leq \Xi, \quad \forall k \geq 0, \tag{12}$$

$$-\dot{\Xi} \leq \frac{\xi_{o,k+1} - \xi_{o,k}}{t_{k+1} - t_k} \leq \dot{\Xi}, \quad \forall k \geq 0. \tag{13}$$

Obviously, the function (11) as well as the constraints (12,13) are *linear* in values of the input sequence evaluated on the time grid: $\xi_{o,k}$. Hence, maximization of (11) with respect to $\xi_{o,k}$ under the constraints (12,13) is LP will deliver the optimal input sequence at times $\{t_k\}$. For practical reasons, the time grid $\{t_k\}$ can only cover a finite interval, say $[0, t_\infty]$. As observed in Sec. 3, the error through finite approximation becomes small if t''_N is sufficiently large; error bounds can be derived easily. For practical implementation, t_∞ should be chosen larger than the largest time constant of the system Π. The Linear Program (11,12,13) over a finite time interval, however, yields the only the "last part" of the optimal input signal, as ξ_o is the *time reversed* input signal. The first part of the optimal input sequence can be constructed as in

Cor. 3. (4.). The worst case output is then obtained by simulating (10).

Remark 3 *The solution outlined so far implicitly assumes, that we exploit some knowledge on the integer k in (10), as indicated in Remark 1. Neglecting this knowledge, we have to solve two liner programs instead.*

5 Multivariable Case

We extend our approach to multivariable systems, i.e. ξ and λ are vector valued signals. What we have in mind is the treatment of multivariable control systems with constraint control signals, i.e. we regard the control signal as output, $\lambda = u$, the reference signal as input, $\xi = r$, and Π is the transfer function defined by $u = \Pi \cdot r = K(I + GK)^{-1} \cdot r$, assuming the standard control control system with controller K and plant G. Therefore, it is useful to restrict the input ξ componentwise, in order to handle each reference channel separately from the others. Hence, the constraints are as follows:

$$|\xi(t)| \preceq \Xi, \ t > 0 \tag{14}$$
$$|\dot{\xi}(t)| \preceq \dot{\Xi}, \ t > 0 \tag{15}$$

and $\xi(t) = 0, t \leq 0$, in complete analogy to Def. 1. Read $\preceq$ as a componentwise $\leq$ and evaluate $|\cdot|$ in this context componentwisely. Consequently, we call the set of all signals ξ fulfilling these constraints $(\Xi, \dot{\Xi})$-admissible, with $\Xi, \dot{\Xi}$ are now being vectors with positive entries. Furthermore, we define the maximum amplitude of the n-dimensional output $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ componentwisely as

$$\Lambda_m := (\Lambda_{1,m}, \ldots, \Lambda_{n,m})^T \tag{16}$$

where $\Lambda_{i,m}$ is defined as in Def. 2 (the input ξ_o in (7) now being a vector valued signal).

The remaining question is, how the solution proposed in Sec. 4 can be used in the multivariable setup. Therefore, we first look onto a system with one output λ and k inputs $\xi = (\xi_1, \ldots, \xi_k)^T \in \mathcal{A}(\Xi, \dot{\Xi})$. Then $\lambda(s)$ is given by

$$\lambda(s) = \Pi_1(s) \cdot \xi_1(s) + \cdots + \Pi_k(s) \cdot \xi_k(s). \tag{17}$$

We abbreviate the response to each of the input channels by $\tilde{\lambda}_i(s) := \Pi_i(s) \cdot \xi_i(s)$. Now we are looking for the maximum output amplitude Λ_m. Using (17), the maximum output amplitude is given by

$$\Lambda_m = \sum_{i=1}^{k} \tilde{\Lambda}_{i,m}. \tag{18}$$

It follows directly, that Λ_m is achieved for a certain *vector* $\xi = (\xi_1, \ldots, \xi_k)^T \in \mathcal{A}(\Xi, \dot{\Xi})$, as all input channels can be chosen independently to maximize their contributions $\tilde{\lambda}_i$ in (18). In the multivariable case with n outputs, we simply apply the first step for each component: according to (16), the components $\Lambda_{i,m}$ of Λ_m can be calculated as in (18). We should, however, note that when using this approach, the maximum output amplitude will not be reached in all channels in one "operation mode". Consider for instance a SIMO system, then the maximum output amplitude of channels i, j may be achieved when feeding the system with certain admissible input signals ξ^i, ξ^j, which are in general different from each other (but still both admissible!). Thus, when feeding the system with input signal ξ^i, output channel i will achieve its maximum amplitude, but $\sup_t \|\lambda^j\| = \sup_t \|\pi^j \star \xi^i\| \leq \sup_t \|\pi^j \star \xi^j\|$. This "overestimation" appears because the definition of the maximum amplitude (16) is *not* a norm, and can therefore not be avoided.

6 Uncertain SISO Systems

Suppose now a set of stable LTI SISO systems with transfer functions

$$\Pi_\theta(s) := \Pi_c(s) + B(s) \cdot \theta, \tag{19}$$

where $B(s) = [B_1(s), \ldots, B_n(s)]^T$ is a set of stable functions, for instance orthonormal basis functions, and $\theta = [\theta_1, \ldots, \theta_n]^T$ is a parameter, located in a rectangular box:

$$\theta_i^{lo} \leq \theta_i \leq \theta_i^{hi}, \quad \forall i. \tag{20}$$

Model set (19) is affinely parameterized in uncertainty, given by (20). Let the input signal ξ obey the same constraints as above, i.e. (1,2). Therefore, finding the maximum output amplitude of system (19) can be solved via Quadratic Programming (QP), when looking at a discretized version: $\Lambda_m = \sup_{\xi_{o,k}} \sum_{k=0}^{\infty} \pi_{\theta,k} \xi_k$, where $\{\pi_{\theta,k}\}$ is the impulse response of system Π_θ in (19), and ξ_k is a (time inverted) admissible input signal, which can be (approximately) described for the discrete time case by

$$-\Xi \leq \quad \xi_{o,k} \quad \leq \Xi, \quad \forall k \geq 0, \tag{21}$$

$$-\dot{\Xi} \leq \frac{\xi_{o,k+1} - \xi_{o,k}}{t_{k+1} - t_k} \leq \dot{\Xi}, \quad \forall k \geq 0. \tag{22}$$

The problem, to be solved on a finite time grid of size, say, $N+1$ is then:

$$\max_{\theta_i,\xi_j}\left\{[\pi_{c,0},\ldots,\pi_{c,N}]\cdot\begin{bmatrix}\xi_0\\ \vdots\\ \xi_N\end{bmatrix}+[\theta_1,\ldots,\theta_n]\cdot\underbrace{\begin{bmatrix}b_{1,0}&\ldots&b_{1,N}\\ \vdots&\ldots&\vdots\\ b_{n,0}&\ldots&b_{n,N}\end{bmatrix}}_{=:b}\cdot\begin{bmatrix}\xi_0\\ \vdots\\ \xi_N\end{bmatrix}\right\}, \quad (23)$$

where $[\pi_{c,0},\ldots,\pi_{c,N}]$ is the impulse response of Π_c, evaluated at the first $N+1$ time stamps, and $b_{i,0},\ldots,b_{i,N}$ is the impulse response of basis function B_i, evaluated on the same time grid. The restrictions on θ_i and ξ_j are given by (20,21,22), which are obviously linear. Denoting $\xi=[\xi_0,\ldots,\xi_N]^T$, likewise $\pi_c=[\pi_{c,0},\ldots,\pi_{c,N}]$ and adding obvious zeros blocks, the above maximization can be written as:

$$\max_{\theta_i,\xi_j}\left\{[0,\pi_c]\begin{bmatrix}\theta\\ \xi\end{bmatrix}+\begin{bmatrix}\theta^T & \xi^T\end{bmatrix}\begin{bmatrix}0&b\\ 0&0\end{bmatrix}\begin{bmatrix}\theta\\ \xi\end{bmatrix}\right\} \quad (24)$$

subject to (20,21,22), We observe that the description of the model uncertainty in (20) can be replaced by any other linear set of parameters without changing the character of the optimization problem.

7 Illustrative Example

We consider an uncertain system as in (19) with $\Pi_c(s)=\frac{s^2+0.4}{s^2+1.4s+1}$ and $B(s)$ a $3rd$ order Laguerre basis with pole $p=-1$. The uncertain parameters obey $\theta_i\in[0.9,1.1], i=1,2,3$. Using a grid of length 177 for the impulse

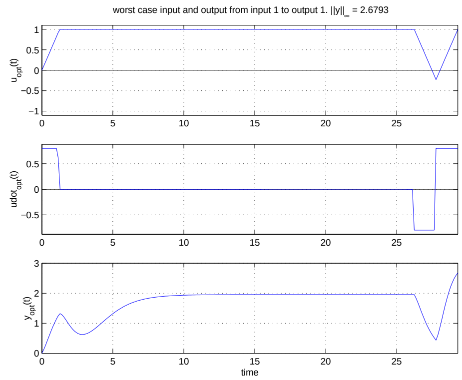

Fig. 2. Model Set: Worst case input, its derivative and worst case output (top-down).

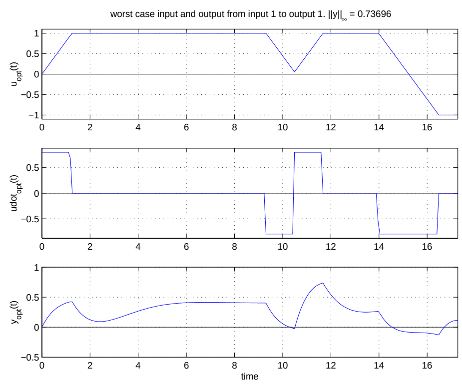

Fig. 3. Nominal model Π_c: Worst case input, its derivative and worst case output (top-down).

responses, the worst case output for all $(1, 0.8)$-admissible input signals is given by $\Lambda_m = 2.68$. Moreover, the "worst case parameter vector" is calculated to $\theta = (1.1, 1.1, 1.1)^T$. All worst case signals are reported in Fig. 2. For comparison, we compute the worst case output of the "nominal" model Π_c, which is significantly lower: $\Lambda_m^{nom} = 0.74$. The signals for this case are depicted in Fig. 3

8 Conclusions and Related Works

We calculated the worst case output amplitude of stable systems, excited with input signals that are bounded in amplitude and rate. In the case of a SISO system, this problem was recasted to a Linear Programming problem: a numerical algorithm was formulated to compute the worst case input and to calculate the worst case output. This is a significant computational simplification compared to the “constructive” solution used so far. The same framework can be applied for multivariable system, when the worst case output is defined componentwisely. A consequence of the Linear Programming approach is that uncertain systems, affinely parameterized in th uncertain parameter vector, can be treated via Quadratic Programming.

Solving this problem is a necessary and important step within several non-conservative controller design procedures for systems with hard bounds on the control signal, as we are now able to calculate the maximum control signal and adapt the controller in such a way, that we meet the prescribed bound on the control signal *exactly* Moreover it enables us to check the maximum amplitude of an arbitrary signal within the control system for an already existing controller. These control applications are presented in detail in [7–9].

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References

[1] D. S. Bernstein and A. N. Michel. A chronological bibliography on saturating actuators. *Int. J. of Robust and Nonlinear Control*, 5:375–380, 1995.

[2] B. Birch and R. Jackson. The behaviour of linear systems with inputs satisfying

certain bounding conditions. *J. of Electronics and Control*, 6:366–375, 1959.

[3] N. Dourdoumas. Prinzipien zum Entwurf linearer Regelkreise mit Beschränkungen. *Automatisierungstechnik*, 35(8):301–309, Aug. 1987.

[4] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust Constrained Model Predictive Control using LMIs. *Automatica*, 32(10):1361–1379, Oct. 1996.

[5] M. V. Kothare, P. J. Campo, M. Morari, and C. N. Nett. A unified framework for the study of Antiwindup designs. *Automatica*, 30(12):1869–1883, Dec. 1994.

[6] R. W. Reichel. *Synthese von Regelsystemen mit Beschränkungen bei stochastischen Eingangsgrössen*. PhD thesis, Dept of EE, University of Paderborn, 33095 Paderborn, Germany, 1984.

[7] W. Reinelt. Robust control of a two-mass-spring system subject to its input constraints. In *Proc. of the ACC*, pp. 1817–1821, Chicago, IL, USA, June 2000.

[8] W. Reinelt. Design of optimal control systems with bounded control signals. In *Proc. of the ECC*, Porto, Portugal, Sept. 2001. Submitted.

[9] W. Reinelt and M. Canale. Robust control of SISO systems subject to hard input constraints. In *Proc. of the ECC*, Porto, Portugal, Sept. 2001. Submitted.

[10] A. Saberi, A. A. Stoorvogel, and P. Sannuti. *Control of Linear Systems with Regulation and Input Constraints*. Springer Verlag, London, UK, 2000.

[11] A. A. Stoorvogel and A. Saberi, guest editors. Special issue on control problems with constraints. *Int. J. of Robust and Nonlinear Control*, 9(10), 1999.

[12] S. Tabouriech and G. Garcia, editors. *Control of Uncertain Systems with Bounded Inputs*, vol. 227 of *LNCIS*. Springer Verlag, London, UK, 1997.

[13] A. R. Teel and N. Kapoor. The $\mathcal{L}_2$ anti-windup problem: its definition and solution. In *Proc. of the ECC*, Brussels, Belgium, 1997.