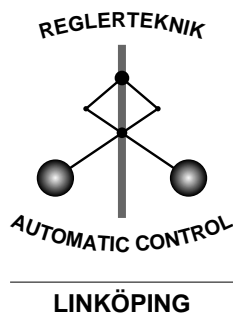


\mathcal{H}_∞ Loop Shaping for Systems with Hard Bounds

Wolfgang Reinelt

Department of Electrical Engineering
Linköping University, S-581 83 Linköping, Sweden
WWW: <http://www.control.isy.liu.se/~wolle/>
Email: wolle@isy.liu.se

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Wolfgang Reinelt

Division of Automatic Control, Dept of Electrical Engineering,
Linköping University, 581 83 Linköping, Sweden,
email: wolle@isy.liu.se, <http://www.control.isy.liu.se/~wolle/>

Abstract

We study multivariable control systems with hard bounded control signals. The reference signals are hard-bounded in their amplitude and speed, which appears in many applications. This description leads therefore to a non-conservative controller design. The design procedure itself is based on \mathcal{H}_∞ Loop Shaping; the problem of systematic adaption on the weights when receiving a too large control signal after a loop shaping step is discussed. Finally, design examples are given.

Keywords: Robust Control, \mathcal{H}_∞ Loop Shaping, Multivariable Systems, Hard Bounds, Saturations.

1 Introduction and Motivation

Most practical control problems are dominated by *hard bounds*. Valves can only be operated between fully open and fully closed, pumps and compressors have a finite throughput capacity and tanks can only hold a certain volume. These input- or actuator-bounds convert the linear model into a nonlinear one. Exceeding these prescribed bounds causes unexpected behaviour of the system – large overshoots, low performance or (in the worst case) instability. Furthermore, process models are always inaccurate – even extremely detailed models may contain unknown or slowly changing physical parameters; so the controller has to manage the difference between the model (used for design) and the real plant. Bridging the gap between model and real plant is the field of *robust controller design*.

\mathcal{H}_∞ Loop Shaping is a popular design method to form the open loop (i.e. its singular values in the frequency range of interest) by introduction of weights, in order to fulfill certain aims as disturbance rejection etc. We shortly state this method in section 2.

We discuss an extension of this procedure in order to meet a prescribed hard bound for the control signal. In the case of more control signals, we are able to handle a hard bound for each of them. The first step is to calculate the maximum of the control signal(s) for a given set of reference signals: the reference signal and its first derivative are bounded. This appears in many systems, for example in a tank, not only the liquid-level is bounded (by the tank's height), additionally the liquid cannot change its level arbitrarily fast. All in all, the componentwise handling of the control signal bounds and the described class of reference signals produce a non-conservative controller. We discuss this in section 3.

One main point is the systematic adjustment of the design weights during the loop shaping procedure in the case that the bounds are not met. We derive an explicit relation between maximum control variable and the singular values of the corresponding transfer function, visible during the loop shaping procedure. The procedure works for LTI multivariable plants and is presented in section 4. The handling of the problem in the discrete time case is briefly outlined in section 5.

Two detailed examples are given in section 6. The first example – the control of the vertical dynamics of an aircraft – demonstrates the usage of the procedure in the multivariable and continuous time case. As a second example, we discuss the two-mass-spring benchmark problem (as posed on the 1992 ACC) in the discrete time domain. We discuss the controller design and analyse robust stability and performance with regard to parameter changes.

2 \mathcal{H}_∞ Loop Shaping Design Procedure – basic facts

As we adapt the \mathcal{H}_∞ Loop Shaping Design Procedure (LSDP) in some way, we present its basic facts first. All results including the proofs and more details can be found in [4]. We use the notation from standard robust control theory textbooks [3].

A continuous time state-space system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

will be denoted (A, B, C, D) or $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$. The corresponding transfer-function is given by $G(s) = C(sI - A)^{-1}B + D$ and will be abbreviated as $G(s) \stackrel{s}{=} (A, B, C, D)$ or $G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$. As "degree" of a transfer-function we will understand the McMillan degree. The ∞ -norm of a transfer-function matrix G is defined by $\|G\|_\infty := \sup_\omega \bar{\sigma}(G(i\omega))$ where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. \mathcal{H}_∞ denotes the space of all real rational transfer-function matrices with finite ∞ -norm and no poles in $Re(s) > 0$. Let P and Q the controllability resp. observability gramian of G . The Hankel singular values of G are given by $\sigma_j := \lambda_j^{1/2}(PQ)$, $j = 1, \dots, n$, where n is the degree of G and λ_j denotes the eigenvalue. The Hankel singular values are ordered by convention: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. The Hankel Norm of G is given by $\|G\|_H := \sigma_1(G)$. Due to Nehari's Theorem, $\|G\|_H \leq \|G\|_\infty$ holds.

Let $G = M^{-1}N$ be the plant to be controlled, represented in the normalized left coprime factorization (NLCF) and G_Δ be a perturbation of this nominal plant. The uncertainty is straightforward modelled as an error of the coprime factors (N, M) :

$$G_\Delta = (M + \Delta_M)^{-1}(N + \Delta_N). \quad (1)$$

For $\epsilon > 0$ a perturbation $\Delta = [\Delta_N, \Delta_M]$ of a transfer function as described in eqn.(1) is called (ϵ -) admissible, when $\|\Delta\|_\infty < \epsilon$ holds. The set of all (ϵ -)admissible perturbations is denoted by \mathcal{D}_ϵ :

$$\mathcal{D}_\epsilon := \{\Delta : \Delta \in \mathcal{H}_\infty; \|\Delta\|_\infty < \epsilon\} \quad (2)$$

This approach has several advantages: the error bounds of the coprime factors can be chosen smaller than those of e.g. a multiplicative error and all participated "components" – numerator, denominator and their errors – are stable even if the plant itself is unstable. Additional advantages become clear when discussing robust stabilization within this framework:

2.1 Robust Stabilization Theorem The controller K stabilizes the uncertain plant $G_\Delta = (M + \Delta_M)^{-1}(N + \Delta_N)$ for all $\Delta = [\Delta_N, \Delta_M] \in \mathcal{D}_\epsilon$ iff it stabilizes the nominal plant G and the following equation holds:

$$\left\| \left\| \begin{bmatrix} K(I - GK)^{-1}M^{-1} \\ (I - GK)^{-1}M^{-1} \end{bmatrix} \right\| \right\|_\infty \leq 1/\epsilon. \quad (3)$$

Moreover, the theorem comes with a calculation of the controller K for a given ϵ , details are given in [4]. It is clear from the Robust Stabilization Theorem 2.1, that we are in principle looking for the largest positive number $\epsilon (= \epsilon_{\max})$, which fulfils equation (3). In this case, we will have "maximal robustness". Because of the NLCF representation, this *maximum stability margin* can be obtained in a surprisingly explicit manner by:

$$\epsilon_{\max} = \left(\inf_{\text{stab } K} \left\| \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}M^{-1} \right\| \right\|_\infty \right)^{-1} = \sqrt{1 - \|[N, M]\|_H^2} \quad (4)$$

We refer to the controller, ensuring the maximum stability margin ϵ_{\max} , as the *optimal* controller. All other controllers producing an $\epsilon < \epsilon_{\max}$ are called *suboptimal* controllers. For the suboptimal controllers we define the *performance factor* f by $f := \epsilon_{\max}/\epsilon$ (> 1). In practice, this performance factor is the design-parameter for the computation of the controller. Because the maximum stability margin ϵ_{\max} is not known a-priori,

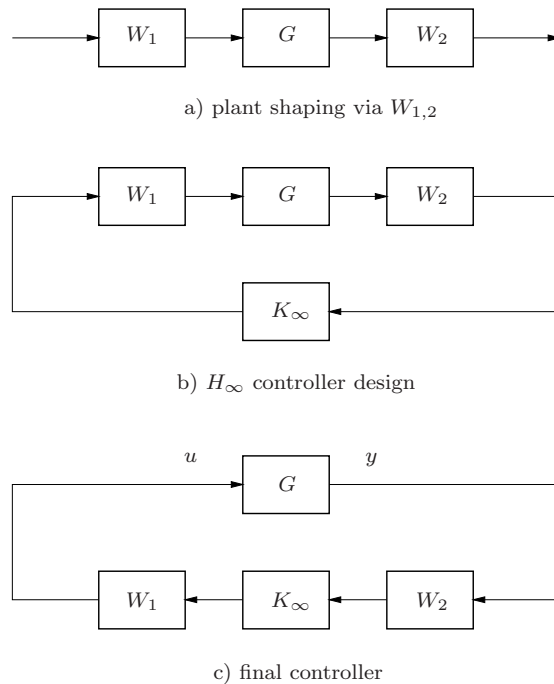


Figure 1: Loop shaping and controller design in three steps.

the designer decides via performance factor f , whether an optimal or a suboptimal controller is wanted. To avoid numerical problems, it is common to design a suboptimal controller with performance factor as $f = 1.1$.

The results until now enable us to robustly stabilize uncertain plants. Performance features as disturbance rejection etc. are incorporated by open loop shaping. The open loop is shaped using input- and output-weights, until a desired shape is reached. A major drawback is inherent in all open loop shaping methods: one can only shape the plant instead of the open loop (consisting of plant *and* controller) and the shape of the open loop has to be checked after the controller design. Within this technique, this problem is solved elegantly: a large stability margin, resulting from the controller design, ensures a similar shape of shaped plant and final open loop. Clear from eqn.(4), that the stability margin ranges in $(0, 1)$. In practice, a stability margin of about $\epsilon \geq 0.3$ is large enough.

All in all, the complete \mathcal{H}_∞ Loop Shaping Design Procedure consists of the following steps (see figure 1):

1. Choose a performance factor f and weights W_1 and W_2 to shape the plant.
2. Controller-design (\mathcal{K}_∞) for the shaped plant and calculation of the stability margin ϵ .
3. Calculation of the final controller (including the weights).
4. Decide whether the design-objectives are fulfilled or not:
 - Is the stability margin ϵ large enough?
 - Are the performance-objectives fulfilled?

If not, choose other weights W_1, W_2 (and/or another performance factor f) and go back to the first step.

3 Maximum Control Signal in a Control System

As motivated in the introduction, we study control systems with reference signals, bounded in amplitude and speed. Aim of the controller design is to handle hard bounds of the control signal. The signals are depicted

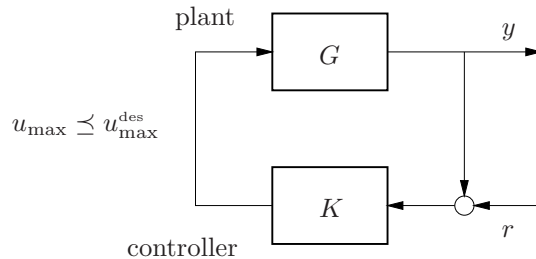


Figure 2: Control loop with constraint control variable u .

in the standard control loop in figure 2. We start with the following definitions, which are extensions of those given by Reichel [6] to the multivariable case, as outlined in [7]:

3.1 Definition (Admissible Reference Signal) Given $0 \preceq R, \dot{R} \in \mathbb{R}^n$. Then a vector-valued reference signal r is called (R, \dot{R}) -admissible, when the following properties hold:

1. $r(t) = 0$ for all $t \leq 0$,
2. $|r(t)| \preceq R$ for all $t > 0$ and
3. $|\dot{r}(t)| \preceq \dot{R}$ for all $t > 0$,

whereas \preceq denotes componentwise \leq and $|\cdot|$ has to be evaluated componentwisely in this context. The set of all (R, \dot{R}) -admissible reference signals is denoted by $\mathcal{A}(R, \dot{R})$.

3.2 Definition (Maximum Control Signal) Given the internally stable standard control loop as in figure 2. We call

$$u_{\max} := \begin{pmatrix} \sup\{\|u_1\|_{\infty}; \forall r \in \mathcal{A}(R, \dot{R})\} \\ \vdots \\ \sup\{\|u_n\|_{\infty}; \forall r \in \mathcal{A}(R, \dot{R})\} \end{pmatrix} \quad (5)$$

the Maximum Control Signal.

The definition of the admissible reference signal is quite straightforward from the motivation. The componentwise definition of the maximum control signal enables us to handle hard bounds for each of the control signals, which is a clear advantage compared to the ∞ -norm of a vector-valued signal, for example. Additionally, we state the following system-theoretic result given by Reichel [6]:

3.3 Theorem and Algorithm (Reichel, 1984) Given a linear and time invariant SISO system with input $x(t)$ and output $y(t)$. Let the input be (X, \dot{X}) -admissible.

- (a) There exists an algorithm that determines the maximum output of this system: $y_{\max} = \sup_{t>0} |y(t)|$ for all admissible reference signals.
- (b) An (X, \dot{X}) -admissible input exists, so that y_{\max} is achieved.

The algorithm outlined in the original work [6] constructs a worst case input x . This solution is also discussed in [9]. An alternative numerical solution of the problem, based on nonlinear optimization, has been shown in [7]. Independent of the numerical solution, Theorem 3.3 can be used in the SISO case to determine the maximum control signal u_{\max} , when the reference signal $r(t)$ is (R, \dot{R}) -admissible (with the obvious replacement r instead of x and u instead of y).

In the following we will see, that it is also useful in the multivariable case. Let us first look onto a system with one output y and k inputs $x = (x_1, \dots, x_k)^T \in \mathcal{A}(X, \dot{X})$. Then $y(s)$ is given by

$$y(s) = H_1(s) \cdot x_1(s) + \dots + H_k(s) \cdot x_k(s) \quad (6)$$

We abbreviate $\tilde{y}_i(s) = H_i(s) \cdot x_i(s)$. Now we are looking for the maximum output amplitude y_{\max} . Using eqn. (6), the maximum output amplitude is given by

$$y_{\max} = \sum_{i=1}^k \tilde{y}_{i,\max}. \quad (7)$$

It follows directly, that y_{\max} is achieved for a certain *vector* $x = (x_1, \dots, x_k)^T \in \mathcal{A}(X, \dot{X})$.

In the multivariable case with n outputs, we simply apply the first step: according to the definition, the components $y_{\max,j}$ of y_{\max} can be computed as in equation (7).

The method presented in this section enables us to calculate the maximum output of a system with certain input restrictions; in particular, it enables us to calculate the maximum control signal of a control system, when the external signal, i.e. the reference signal fulfills these constraints. We will exploit this in the next section.

4 \mathcal{H}_∞ Loop Shaping for Systems with Constraints

Given a (multivariable) plant G , restrictions $R, \dot{R} \succeq 0$ for the reference signal and a desired maximum control signal $u_{\max}^{\text{des}} > 0$ we extend the Loop Shaping Procedure (see sec. 2, figure 1) in the following way:

1. Choose a performance factor f and weights W_1 and W_2 to shape the plant.
2. Controller-design for the shaped plant and calculation of the stability margin ϵ .
3. Calculation of the final controller (including the weights).
4. Decide whether the design-objectives are fulfilled or not:
 - Is the stability margin ϵ large enough?
 - Are the performance-objectives fulfilled?
 - Does $u_{\max} \leq u_{\max}^{\text{des}}$ hold?

If not, choose other weights W_1, W_2 (and/or another performance factor f) and go back to the first step.

Theorem 3.3 enables us to determine the maximum control signal u_{\max} within the control loop, which allows the check for the desired bound on the control signal in step 4. This additional check is the only difference to the classical LSDP. We refer to this procedure in the following with Extended Loop Shaping Design Procedure (ELSDP).

Finally, we discuss a question we dropped when repeating the classical LSDP: after a first choice of weights, our design objectives are usually *not* fulfilled – the weights have to be adjusted. In the case of a too small stability margin ϵ the strategy is clear from the classical LSDP: because there is no explicit relation known between achieved stability margin and weights, we have to examine the singular values of the shaped plant and the achieved open loop. In the frequency range with a significant difference, our weights are incompatible with the plant and have to be adjusted. In the following we discuss the remaining open question: Is there a strategy for correct and systematic adjustment of the weights, when the control signal bound fails after a loop shaping step?

Within loop shaping, we work on the singular values of different interesting transfer functions. Thus we are searching for a relation between the singular values of the transfer function and the maximum control signal u_{\max} . The relation between reference signal and control signal is given by

$$\begin{aligned} u(s) &= H(s) \cdot r(s) \quad \text{resp.} \\ u(t) &= h(t) \star r(t) \end{aligned}$$

where " \star " denotes the convolution. Because of the componentwise definition of the maximum control variable, we restrict our following examinations to the case of a single control variable. The generalization follows immediately as in the previous section (by componentwise usage).

Well known from linear system theory is the following inequality

$$\|u\|_\infty \leq \|h\|_1 \|r\|_\infty, \quad (8)$$

where $\|h\|_1$ denotes the 1-norm of the impulse response $\|h\|_1 := \int_0^\infty |h(t)| dt$. To get the link from the (time domain) impulse response to the (frequency domain) transfer function, we state a result from model reduction [2]:

4.1 Lemma (Glover, Curtain and Partington, 1985) Let H be a asymptotically stable and strictly proper transfer function, then

$$2 \cdot \|H\|_N \geq \|h\|_1 \geq \|H\|_\infty \quad (9)$$

holds whereas $\|\cdot\|_N$ denotes the nuclear norm $\|H\|_N := \sum_{j=1}^n \sigma_j(H)$ and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ are the Hankel singular values in descending order.

Combining equation (8) and (9) we have the following situation:

$$\|u\|_\infty \leq \|h\|_1 \cdot \|r\|_\infty \leq 2 \cdot \|H\|_N \cdot \|r\|_\infty. \quad (10)$$

Exploiting $\sigma_1(H) = \|H\|_H \leq \|H\|_\infty$, the previous equation (10) degenerates to

$$\|u\|_\infty \leq 2n \cdot \|H\|_\infty \cdot \|r\|_\infty \quad (11)$$

where n denotes the McMillan degree of H .

We now turn back to our final aim: the relation between the singular values of H and $\|u\|_\infty$. As the control loop is internally stable, the transfer function H is stable. Following equation (11), we see that decreasing the ∞ -norm of $H(s)$ decreases an upper bound for the maximum control signal.

Suppose, the maximum control signal is too high after a loop shaping step. We then have to decrease the maximum singular value of H in the frequency range where the ∞ -norm appears. In the case of a too low maximum control signal, we have to increase the maximum singular value in that frequency range. We point out, that this affects only an *upper bound* for the maximum control signal. This practical value of this guideline is shown in the examples.

The last question to answer is, if a relation between the singular values of the weights and those of the transfer function H exists. Using the relations $H(s) = (I - G(s)K(s))^{-1} K(s)$ and $K(s) = W_1(s)K_\infty(s)W_2(s)$, from the loop shaping procedure, we get

$$\begin{aligned} \bar{\sigma}(H) = \bar{\sigma}((I - GK)^{-1}K) &\leq 1/\underline{\alpha}(I - GK) \bar{\sigma}(K) \\ &\approx \bar{\sigma}(K) \quad (\text{at freq. with } \bar{\sigma}(GK) \ll 1) \end{aligned} \quad (12)$$

$$\begin{aligned} &= \bar{\sigma}(W_1 K_\infty W_2) \\ &\leq \bar{\sigma}(W_1) \bar{\sigma}(K_\infty) \bar{\sigma}(W_2). \end{aligned} \quad (13)$$

Obvious from the last inequality (13), decreasing the weights decreases the transfer function H . As the ∞ -norm of H appears at high frequencies (where the open loop gain is low), the restriction on certain frequencies (used in step 12) causes no trouble.

We solved the problem of correct weight-adjustment in the face of a too high maximum control signal. An initial choice of the weights should attack the general shape of the open loop and is discussed in context with original Loop Shaping Design Procedure by McFarlane & Glover [4] and textbooks on robust control, e.g. [3].

5 Discrete Time Case

In order to handle discrete time problems in the same framework, we map the z transfer function to the so-called q -domain by the transformation [1]:

$$G^\#(q) := G^*(z) \Big|_{z = \frac{1+q\frac{T}{2}}{1-q\frac{T}{2}}} \quad (14)$$

(where T is the sampling time). The q transfer functions has the same properties concerning stability as the s transfer function in the continuous time domain, so we are able to apply the LSDP directly in the q -domain. An equivalent result to Reichel's Theorem 3.3 in the z -domain was derived by Peng [5]: for discrete time SISO system with input r fulfilling $r(k) = 0$ ($k < 0$), $|r(k)| \leq r_{max}$ and $|r(k) - r(k+1)| \leq \Delta r_{max}$ ($k \geq 0$) the output maximum $u_{max} := \sup_{k \geq 0} |u(k)|$ can be calculated.

Thus the ELSDP, formulated in section 4, can be applied in the q -domain, with a check for the maximum control signal in the z -domain.

6 Illustrative Examples

6.1 Vertical Plane Dynamics Control of an Aircraft

We study a multivariable, continuous time plant, an aircraft model, which also examined by McFarlane & Glover [4] and others. The model has three inputs and five states. The first three of them are the plant outputs. Its state space model is given by

$$G(s) \stackrel{s}{=} \left[\begin{array}{ccccc|ccc} 0 & 0 & 1.1320 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 & -0.1200 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.0130 & 4.4190 & 0 & -1.6650 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 & 1.5750 & 0 & -0.0732 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad (15)$$

Throughout the example, we determine suboptimal controllers ($f = 1.1$). McFarlane & Glover state the usual performance and stability objectives and the following limits on the control signal $u(t)$:

$$|u_1(t)| < 40, \quad |u_2(t)| < 10, \quad |u_3(t)| < 40. \quad (16)$$

To solve this problem with the ELSDP, we restrict the reference signal by the following values:

$$R = [1, 1, 1]^T, \quad \dot{R} = [5, 11, 3]^T. \quad (17)$$

Analysis of the McFarlane & Glover design

One of the McFarlane & Glover designs [4, section 7.4, design (2)] works with the following diagonal weight $W(s)$:

$$\begin{aligned} w_c(s) &= \frac{s + 0.4}{s} \\ w_1(s) = w_3(s) &= 24 \cdot w_c \\ w_2(s) &= 12 \cdot w_c \\ W(s) &= \text{diag}\{w_1(s), w_2(s), w_3(s)\}. \end{aligned}$$

The usage of *one diagonal weight* ensures a better oversight during the LSDP. Therefore, we restrict ourselves to this class of weights. The gain increases the open loop and thus increases the *0dB* crossover frequency. The integral action will improve the low frequency performance. Looking at the singular values of the unshaped

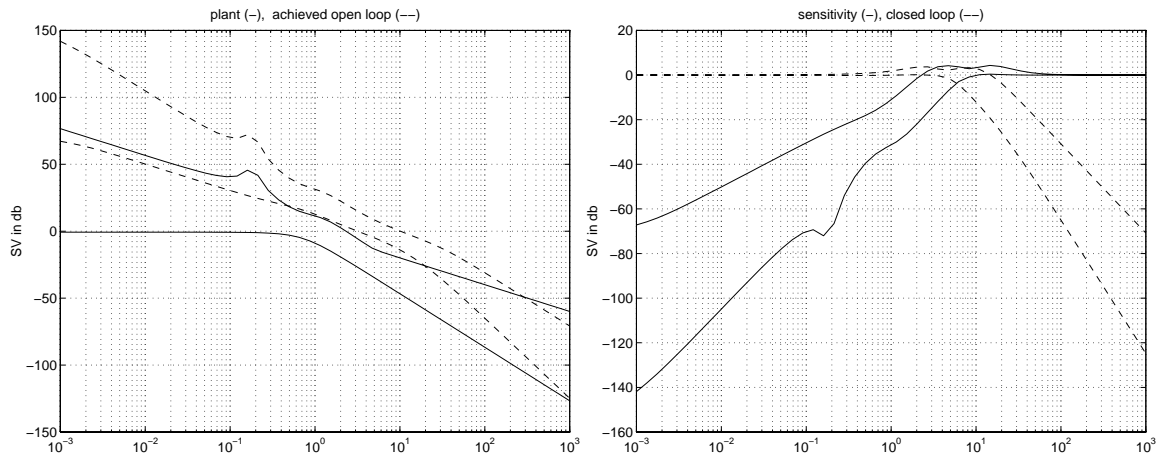


Figure 3: McFarlane & Glover design: singular values of plant and achieved open loop (left) and of sensitivity and closed loop (right).

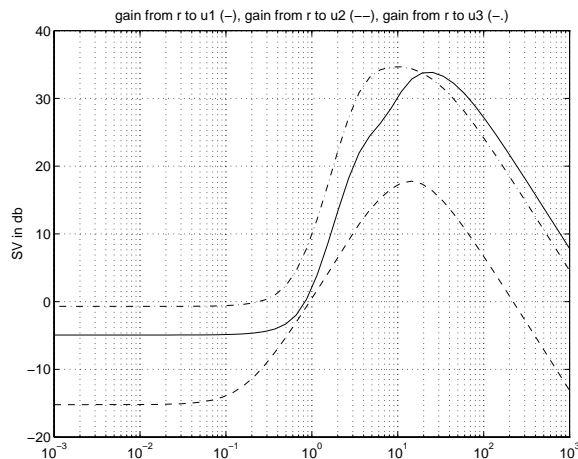


Figure 4: McFarlane & Glover design: singular values for the components of the control variable.

plant (see figure 3-left), the zeros at -0.4 limit the integrator to the low frequency range, so that a too high roll-off rate near the crossover frequency is prevented. This would cause poor robustness properties (i.e. a small stability margin) or even instability (known from Bode's Gain-Phase relations). Starting with this McFarlane & Glover weight for a first study we get the following result for the maximum control signal:

$$u_{\max} = [32.1605, 10.9964, 59.6279]^T$$

and achieve a stability margin of $\epsilon = 0.3444$. Figures 3 and 4 show the singular values¹. Our aim $|u_3(t)| < 40$, failed clearly, the value for $u_{2,\max}$ too large, as well.

Adjustment of the weights

We adjust the weight from the step above in order to achieve the desired maximum control variable. According to the results derived in section 4, we are looking at the singular values of H in figure 4. We see, that they are too high for the frequencies $[1.5, 100]$. Therefore, we decrease the weights w_2 and w_3 in this

¹Throughout this example, we only show the maximum and the minimum singular values due to a better overview.

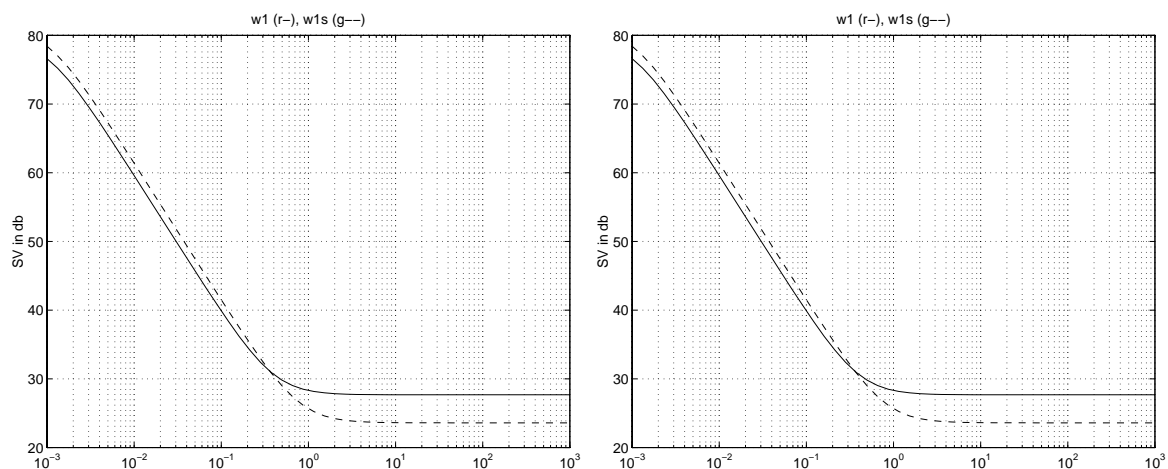


Figure 5: Adjustment of the weights: singular values of the diagonal entries w_1 and w_2 .

frequency range and work with the following weight W_s , see figure 5:

$$\begin{aligned}
 w_{1s} &= \frac{15.13s + 11.79}{s + 0.001} \\
 w_{2s} &= \frac{6.44s + 4.661}{s + 0.001} \\
 w_{3s} &= \frac{14.63s + 11.39}{s + 0.001} \\
 W_s &= \text{diag}\{w_{1s}, w_{2s}, w_{3s}\}
 \end{aligned}$$

Using this new weight, we get the following result on the maximum control signal:

$$u_{\max} = [35.2007, 8.5889, 31.4148]^T$$

and achieve a stability margin of $\epsilon = 0.3205$.

Our design objectives regarding the constraint control variable are fulfilled, figures 6 shows the singular values for the final controller design. The singular values of the transfer functions to the single components of the control signal, depicted in figure 7, have been decreased in the interesting frequency range.

During these designs, the following effect has been observed: comparing two controllers K_i with different performance factors f_i (but produced by the same weights), the resulting maximum control signals fulfilled $u_{2,\max} < u_{1,\max}$ (for $f_1 < f_2$). In general, this would be a useful feature within the procedure. When a large stability margin was achieved, but the meeting of the bounds fails "slightly". Increasing the performance factor would decrease the maximum control signal (and of course decrease the stability margin slightly, too) and the problem is solved without new adjustment of the weights.

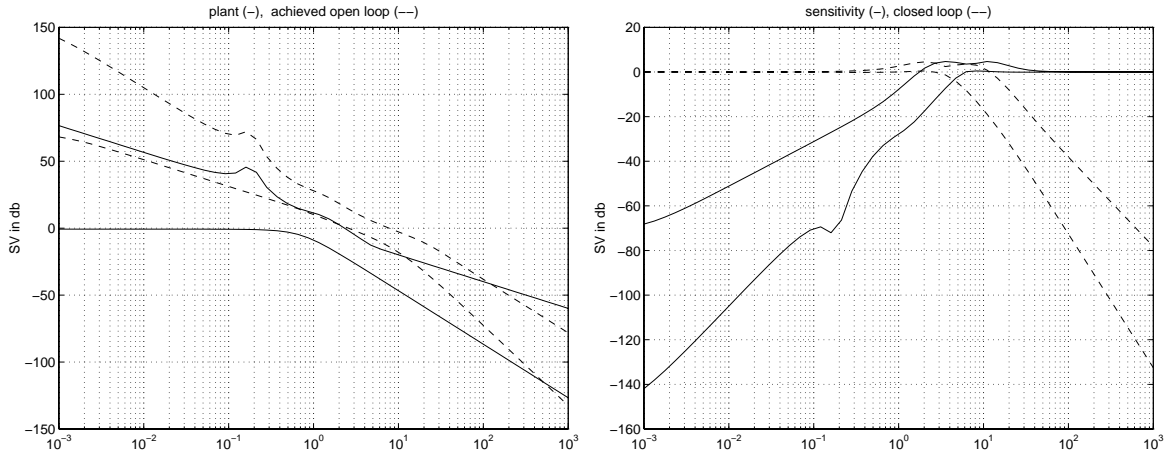


Figure 6: Design with adjusted weights: singular values of plant, achieved open loop, sensitivity and closed loop.

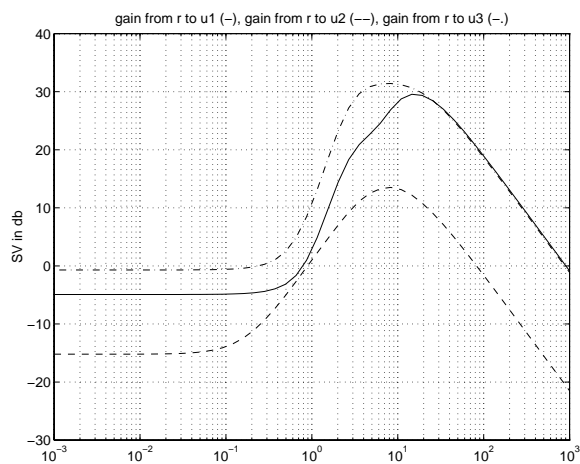


Figure 7: Design with adjusted weights: singular values for the transfer functions to the components of the control variable.

6.2 Two Mass Spring Benchmark Problem

The following problem is a popular problem for robust controller design, discussed for example on the 1992 ACC and described in [10]. The designs presented here are due to [11].

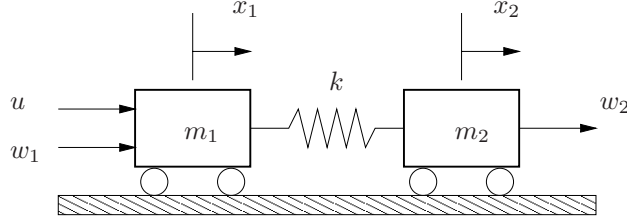


Figure 8: Two mass spring system with uncertain parameters.

Consider the two mass spring system shown in figure 8. It is assumed that for the nominal system $m_1 = m_2 = 1$ and $k = 1$ hold. The control force u acts on body 1, while the position of body 2 is measured. The system can be represented in the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} (u + w_1) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w_2 \quad (18)$$

$$y = x_2 \quad (19)$$

or as transfer-function:

$$G(s) = \frac{k}{m_1 m_2 s^4 + k(m_1 + m_2) s^2} = \frac{y(s)}{u(s)} \quad (20)$$

The aim of one of the problems posed in the benchmark collection [10, problem # 4], is to design a linear feedback-controller with constant gain, with the following properties of the control system:

- A1** control signal for unit step output command tracking is limited to $|u| \leq 1$
- A2** performance: settling time and overshoot are both to be minimized
- A3** robustness: performance and stability with respect to the three uncertain parameters are both to be maximized

The problem is discussed in the discrete time domain. Using a sample time of $T = 0.1s$, the transfer function in the q -domain using transformation (14) is

$$G^\#(q) = \frac{0.0001q^3 - 0.0017q^2 - 0.0502q + 1.0033}{q^4 + 2.0067q^2}.$$

To solve the problem with the ELSDP, we replace property A1 by

- A1'** control signal must be $|u(k)| < 1$ for all $k \geq 0$ for any reference signal with

$$|r(k)| \leq 1 \text{ for all } k \geq 0 \text{ and } |r(k) - r(k-1)| \leq 1 \text{ for all } k \geq 1 \quad (21)$$

In contrast to the first example, we have a *parametric* or *structured* uncertainty. Therefore, we analyse our results looking at variations of the three uncertain parameters k , m_1 and m_2 . One popular method doing this is μ analysis. Indeed, we studied this problem in the continuous time domain using this method [8]. In the q -domain, the structure of the plant is more complicated due to the two transformations, therefore we examine stability and maximum control signal by variation of the parameters. The result of this analysis is therefore a "stability"-interval and a "control signal sufficiently small"-interval for the examined parameter.

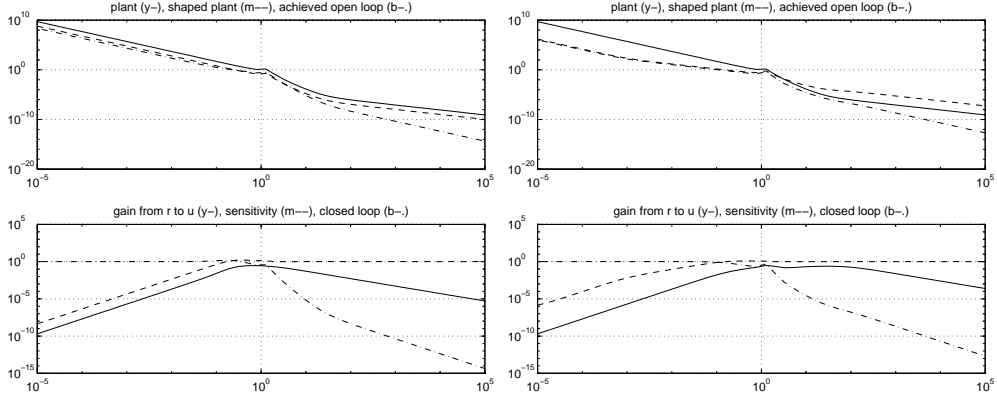


Figure 9: Bode plots for designs #0 and #2.

Controller Design

As in the first example, we choose the Performance Factor $f = 1.1$. We start with a constant weight $w = 0.1775$ and receive a stability margin of $\epsilon = 0.3619$ and a maximum control signal $u_{max} = 0.9973$. This result fulfils property A1' for the nominal case, but analysis of the control system shows, that slight changes of the parameters k , m_1 and m_2 will lead to control signals > 1 . Assume $m_1 = m_2 = 1$ then the system is stable for $k \in [0.5414, 1.0229]$, but A1' is fulfilled in the unsatisfactory small interval $k \in [0.9985, 1.0229]$. Therefore, we decrease the weight to $w_{const} = 0.125$ receiving

$$\begin{array}{ccc} \epsilon & u_{max} & \text{controller } K(z) \\ 0.3684 & 0.6347 & \frac{-0.0233z^4 + 0.0460z^3 + 0.0002z^2 - 0.0460z + 0.0231}{z^4 - 3.7707z^3 + 5.3486z^2 - 3.3804z + 0.8027} \end{array}$$

This design will be quoted as design #0. Bode plots are shown in fig 9 (left). Using this weight as an initial weight for higher order designs, we increase the gain it in the frequency range $[1, 100]$ (in order to achieve a higher maximum control signal and a higher stability margin) using a second order weight. The Bode plots for design #2 are shown in fig 9 (right):

$$\begin{array}{ccc} \epsilon & u_{max} & \text{weight } w_2(z) \\ 0.6290 & 0.9885 & \frac{9.5037z^2 - 15.6981z + 6.1947}{z^2 + 0.3110z - 0.2130} \end{array}$$

Simulation results

In the following, we are going to simulate the two systems with regard to parameter changes. Remarkable, that (in contrast to a series of other works) we have no problems with a too large control signal when changing the parameters. The price we pay for this feature is of course a slower control system. First, we simulate the step responses of the nominal systems, depicted in fig. 10. The two controllers produce a quite different behaviour, design #2 is successful in minimizing the overshoot because it damps the resonance, inherent in the system.

Variation of spring constant k

Table 1 below shows the allowed intervals for k , when $m_1 = m_2 = 1$ are assumed to keep their nominal values. We expect in some meaning from the different stability margins, that the stability interval for design #2 is larger than for design #0, which is not falsified in this experiment. In the case of design #0, the range of a sufficiently small control signal is smaller than the stability interval. Fig. 11 shows the step response for $k = 1.15$, which is out of the stability range for design design #0. Obviously, a step input is sufficient for a unbounded output in this control system.

$m_1 = m_2 = 1$		stability for $k_{min} \leq k \leq k_{max}$		$ u \leq 1$ for $k_{min} \leq k \leq k_{max}$	
weight	ϵ	k_{min}	k_{max}	k_{min}	k_{max}
w_{const}	0.3684	0.4289	1.0176	0.4825	1.0176
w_2	0.6290	0.3734	1.5853	0.3734	1.5853

Table 1: Variation of spring constant k .

Variation of masses m_1 and m_2

Variation of the parameters m_1 and m_2 assuming constant $k = 1$ is depicted in table below. Fig. 12 shows selected simulations for $k = 1$ and $m_1 = m_2 = 2$. In tendency the same results appear in the case $k = 1$ and $m_1 + m_2 = 2$. Again,, in both cases the interval of stability is larger than the interval of a sufficiently small control signal.

$k = 1$ $m_1 = m_2$		stability for $m_{1,min} \leq m_1 \leq m_{1,max}$		$ u \leq 1$ for $m_{1,min} \leq m_1 \leq m_{1,max}$	
weight	ϵ	$m_{1,min}$	$m_{1,max}$	$m_{1,min}$	$m_{1,max}$
w_{const}	0.3684	0.9828	2.5864	0.9828	2.5864
w_2	0.6290	0.6449	2.9027	<i>0.9510</i>	2.9027
$m_2 = 2 - m_1$					
w_{const}	0.3684	0.8685	1.1315	0.8685	1.1315
w_2	0.6290	0.3924	1.6076	<i>0.4924</i>	1.6076

Table 2: Variation of the masses $m_1 = m_2$ resp. $m_1 = 2 - m_2$.

The study of the two mass spring problem shows the usage of the ELSDP in the discrete time domain. We analysed robustness due to parameter changes. However, the treatment of multivariable problems turns out to be difficult in this framework, because due to the transformations from q -domain to z -domain and back, numerical difficulties appear. Sometimes, slightly damped system poles are mapped to unstable ones in the z -domain. Therefore, future directions tend to apply the loop shaping directly in the z -domain (instead of the q -domain).

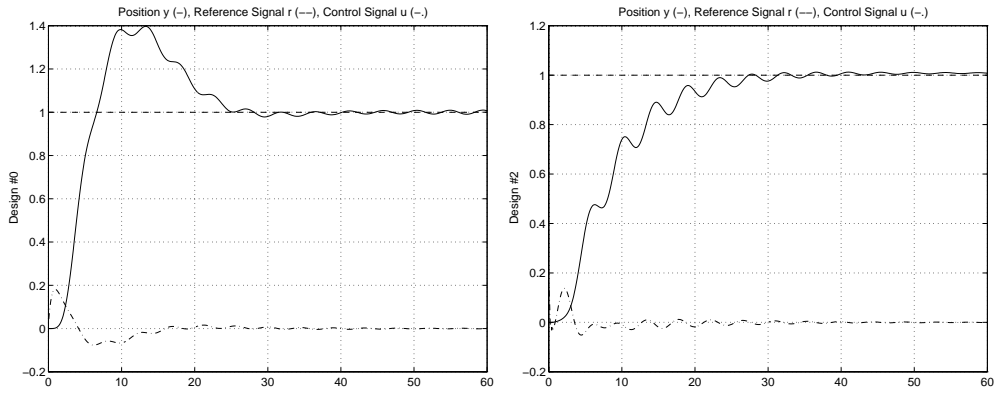


Figure 10: Step responses for $k = m_1 = m_2 = 1$.

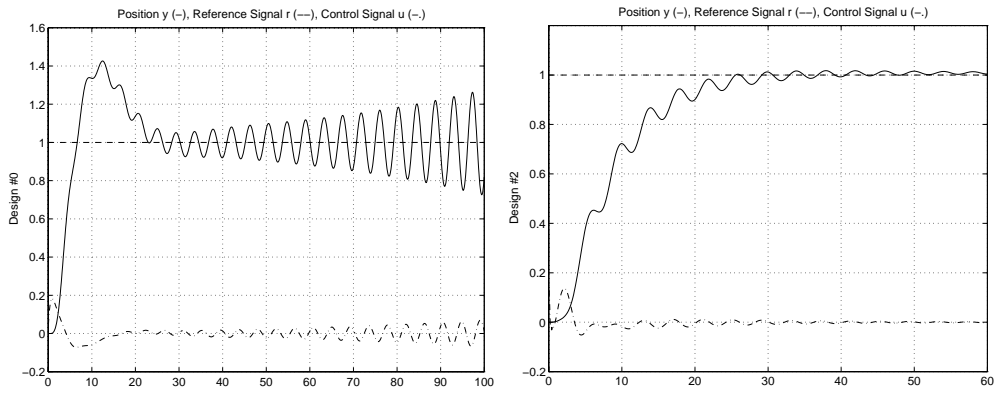


Figure 11: Step responses for $k = 1.15, m_1 = m_2 = 1$.

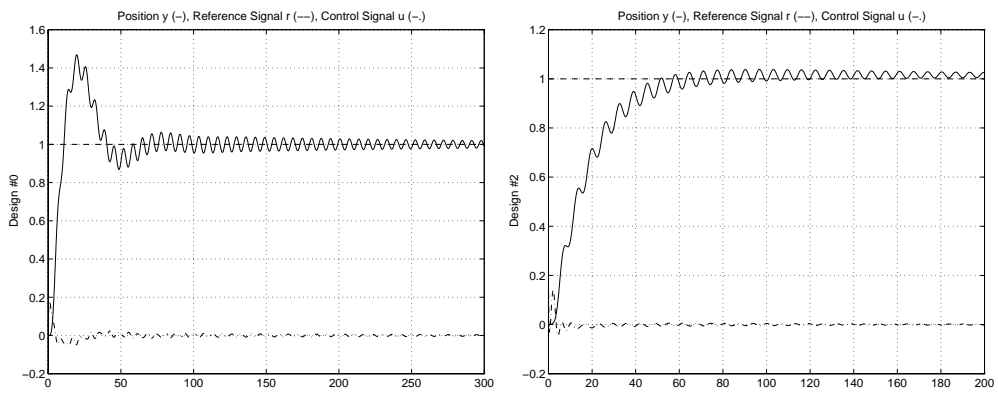


Figure 12: Step responses for $k = 1, m_1 = m_2 = 2$.

7 Conclusions

We studied the control of multivariable control systems with hard bounded control signals. One main point within the extension of the \mathcal{H}_∞ Loops Shaping was the calculation of the maximum control signal for the set of admissible reference signals. The other main point was the systematic adaption on the weights with respect to the control signal. Examples were given for the continuous and discrete time case. Numerical problem appear in the discrete time and multivariable case. Therefore, future reasearch directions will study the loop shaping directly in the z -domain instead of the q -domain. Moreover, the following problems will be studies in future: constraints for other signals than the control signal, speed constraints for the control signal (important for flight control applications for example) and applications with mixed hard and soft bounds.

Acknowledgement

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