Maximum Output Amplitude of Linear Systems for certain Input Constraints

(revised and updated version)

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Maximum Output Amplitude of Linear Systems for certain Input Constraints*

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Abstract

We determine the maximum output amplitude of a system, when the input is bounded by certain constraints. In particular, amplitude and rate of change (i.e. the first derivative) have to be bounded. We show properties of the worst case input and present an algorithm that allows construction of this input and calculates the maximum amplitude of the output. The solution of this problem is a necessary and important step within a couple of recently developed controller-design procedures, dealing with plants with hard-bounded inputs. Nevertheless, it is interesting as a system theoretic task itself and therefore stated separately.

Keywords: Hard Constraints, Saturations, Rate Constraints, Maximum Output Amplitude, Linear Systems.

1 Introduction and Motivation

Most practical control problems are dominated by *hard bounds*. Valves can only be operated between fully open and fully closed, pumps and compressors have a finite throughput capacity and tanks can only hold a certain volume. These input- or actuator-bounds convert the linear model into a nonlinear one. Exceeding these prescribed bounds causes unexpected behavior of the system – large overshoots, low performance or (in the worst case) instability.

Controller design for systems with hard constraints is a vivid area of research, see for example the recent textbook [8] or the overview paper [1] and the references therein. A quite general and unified description of the so-called Anti Windup schemes is given for instance in [2]. Analysis of constraint systems in terms of stability, controllability and feasibility is of interest as well [7, 9, 10, 11].

To solve the constraint control problem in a linear framework, one implicitly has to restrict the amplitude of all external signals – independent from the technique used in particular. A couple of approaches, however, differ from the ones cited above by imposing an additional restriction on the rate of the external signals. In many practical situations, this is a very accurate description of those external signals, *possibly* applied to the control system. In the example of the tank from above, not only the liquid-level is bounded (by the tanks height), additionally the liquid cannot change its level arbitrarily fast. A design, directly based on this description will avoid a conservative control system. In particular, design of optimal controllers has been considered [6] as well as uncertain multivariable systems [4, 5] or systems with process noise under certain statistical assumptions [3].

A common feature of the design procedures described in [3, 4, 6, 5] is, that they all rely *heavily* on the computation of the maximum amplitude of certain signals within the control system (control signal and/or error signal), when the external signal is bounded in amplitude and rate. The underlying idea in an iterative schemes is then to adapt the controller in a certain way, when having calculated the maximum control signal

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exactly (it might be too high or too low) in order to meet the *prescribed* bounds on the control signal for instance. This iterative scheme could be user-interactive, i.e. of a-posteriori character as in [4, 5] or fully automated within an optimization procedure, cf. [6]. However, the core problem in calculating the maximum amplitudes is like this: given a transfer function (here: from reference signal to control signal for instance) and the bounds on amplitude and rate of the input signal (here: reference signal). Calculate the maximum possible output amplitude (here: maximum control signal) for all admissible inputs. This "computational" problem will be solved here. However, the solution is based on constructing the so-called worst case input (that one, which produces the maximum output amplitude for some time stamp), for which we shall show some properties first. Thus, as a by-product of this work, we will not only be able to calculate the maximum output amplitude, but also the worst case input itself. This will give further insight in run-time behavior of constraint control systems.

This work is organized as follows: Section 2 defines the problem, which will then be solved in Section 3. Section 4 presents different possibilities of the numerical solution. The approach is extended to the multi-variable case in Section 5, suited for multivariable control systems. Section 6 illustrates the theory with an example. The work is summarized in Section 7.

2 Problem Statement

We examine a linear and time invariant stable system, which is represented by its transfer function $\Pi(s)$ resp. its impulse response $\pi(t)$. We postpone the extension to the multivariable case to section 5 and concentrate on the SISO case. The input is denoted by ξ, the output by λ. The following constraints hold for the continuous and piecewise[1] differentiable input signal ξ:

$$|\xi(t)| \leq \Xi \tag{1}$$
$$|\dot{\xi}(t)| \leq \dot{\Xi} \tag{2}$$

for $t > 0$, where $\Xi, \dot{\Xi} > 0$ are given constant values and

$$\xi(t) = 0, \quad t \leq 0. \tag{3}$$

We call those reference signals, which fulfill eqns.(1-3) $(\Xi, \dot{\Xi})$-*admissible*, or short $\xi \in \mathcal{A}(\Xi, \dot{\Xi})$. We are looking for the maximum amplitude $\Lambda_m(t)$ of the output λ (up to time t) for all $(\Xi, \dot{\Xi})$-admissible inputs, i.e.

$$\Lambda_m(t) := \sup_{\xi \in \mathcal{A}(\Xi,\dot{\Xi})} \sup_{0<\tau\leq t} |\lambda(\tau)| = \sup_{\xi \in \mathcal{A}(\Xi,\dot{\Xi})} \sup_{0<\tau\leq t} |\pi(\tau) \star \xi(\tau)|, \tag{4}$$

where "$\star$" is the convolution: $\pi(t) \star \xi(t) = \int_0^t \pi(\tau)\xi(t-\tau)d\tau$. Well-known from linear system theory is, that for systems with the only input constraint (1), the maximum output amplitude is given by $\max_{t\geq 0} |\lambda(t)| = \Xi \int_0^\infty |\pi(\tau)| d\tau$, produced by the so-called bang-bang input:

$$\xi(t-\tau) = \Xi \cdot \mathrm{sign}(\pi(\tau)). \tag{5}$$

Thus the problem is trivial unless the additional constraint (2) is imposed.

3 Properties of the Worst Case Input

We now turn to the construction of the maximum output amplitude as stated in eqn.(4). We show some properties of the input signal ξ, which produces the output with the maximum output amplitude. In the following, we call this input signal the *worst case input*. This strategy is motivated by the existence of a worst case input in the simple case in eqn.(5). Let, for a certain time stamp t, the output be given by convolution:

$$\lambda(t) = \int_0^t \pi(\tau)\xi(t-\tau)d\tau =: \int_0^t \pi(\tau)\xi_t(\tau)d\tau, \tag{6}$$

[1] a countable number of time stamps t, where ξ is non-differentiable, is allowed.

where we abbreviate the time inverted input signal by $\xi_t(\tau) := \xi(t-\tau)$. This has the following consequences for the constraints (1-3):

$$\begin{aligned} |\xi_t(\tau)| &\leq \Xi, \quad \tau < t &(7)\\ |\dot{\xi}_t(\tau)| &\leq \dot{\Xi}, \quad \tau < t &(8)\\ \xi_t(\tau) &= 0, \quad \tau \geq t &(9) \end{aligned}$$

3.1 Lemma The function $\Lambda_m(t)$ is monotone increasing (not decreasing) in t. Therefore, the maximum amplitude as defined in eqn.(4) appears for $t \to \infty$, thus $\Lambda_m = \lim_{t\to\infty} \Lambda_m(t)$ is the maximum output amplitude.

Proof. Let $t_0 > 0$ and $\xi_{t_0} \in \mathcal{A}(\Xi, \dot{\Xi})$ an input[2] that produces the maximum amplitude $\Lambda_m(t_0)$. For $t_1 > t_0$ define $\xi_{t_1}(\tau) = \xi_{t_0}(\tau)$, $0 \leq \tau \leq t_0$ resp. $\xi_{t_1}(\tau) = 0$, $t_0 < \tau \leq t_1$. Clearly $\xi_{t_1} \in \mathcal{A}(\Xi, \dot{\Xi})$ and from eqn.(6) follows $\sup_{0<\tau\leq t_1} |\lambda(\tau)| = \sup_{0<\tau\leq t_0} |\lambda(\tau)|$ and thus $\Lambda_m(t_1) \geq \Lambda_m(t_0)$. □

3.2 Definition Suppose there exists an input $\xi_{\infty,o} =: \xi_o$ with maximum output amplitude Λ_m. Then $-\xi_o$ produces the maximum output amplitude Λ_m too. For one of them, say ξ_o, holds

$$\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau \geq 0, \tag{10}$$

i.e. the absolute value in eqn.(4) is obsolete. In the following, we construct this worst case input that produces this maximum output amplitude according to eqn.(10).

3.3 Algorithm (Construction of an auxiliary input) Let $\xi \in \mathcal{A}(\Xi, \dot{\Xi})$ be an arbitrary admissible input. We construct an auxiliary input ξ_H for ξ uniquely by the steps given below (figure 1 illustrates the construction). The set of all possible auxiliary inputs (i.e. all signals with the same properties) is denoted by $\mathcal{A}_H(\Xi, \dot{\Xi})$.

1. Let t_i, $i = 1, \ldots, N$ the zeros of π. Define $\xi_H(t_i) = \xi(t_i)$.
2. If $\pi(t) > 0$ in (t_i, t_{i+1}), let $\dot{\xi}_H(t) = +\dot{\Xi}$ in the neighborhood of t_i and $\dot{\xi}_H(t) = -\dot{\Xi}$ in the neighborhood of t_{i+1}. In the case that this definition leads to the non-unique situation that the two 'slopes' intersect in some $t_* \in (t_i, t_{i+1})$, let $\dot{\xi}_H(t) = +\dot{\Xi}$ in $[t_i, t_*]$ resp. $\dot{\xi}_H(t) = -\dot{\Xi}$ in $[t_*, t_{i+1}]$. Finally, define $\xi_H = \min\{\xi_H, +\Xi\}$.
3. If $\pi(t) < 0$ in (t_i, t_{i+1}), do as in step 2. but with changed signs for $\dot{\xi}_H$ and resulting obvious modifications.
4. Choose $|\dot{\xi}_H(t)| = \dot{\Xi}$ for large times t so that $\lim_{t\to\infty} \xi_H(t) = 0$ in order to fulfill eqn. (9).

3.4 Corollary The following properties of the auxiliary input are clear by construction:

1. $\mathcal{A}_H(\Xi, \dot{\Xi}) \subset \mathcal{A}(\Xi, \dot{\Xi})$, i.e. an auxiliary input is also admissible.
2. Two different inputs $\xi_1, \xi_2 \in \mathcal{A}(\Xi, \dot{\Xi})$ with $\xi_1(t_i) = \xi_2(t_i)$ have the same auxiliary input.
3. $\xi_H(t) \geq \xi(t)$ for $\pi(t) \geq 0$ and $\xi_H(t) \leq \xi(t)$ for $\pi(t) \leq 0$.
4. Fix an admissible input ξ and suppose an arbitrary admissible signal $\xi^* \neq \xi_H$ with property 3, then $\int_0^\infty \pi(\tau)\xi_H(\tau)d\tau > \int_0^\infty \pi(\tau)\xi^*(\tau)d\tau$.
5. The maximum width of the pulses of $\dot{\xi}_H$ in Algorithm 3.3 is given by $T = 2 \cdot \Xi/\dot{\Xi}$.

3.5 Theorem The worst case input is auxiliary input: $\xi_o \in \mathcal{A}_H(\Xi, \dot{\Xi})$.

[2] uniqueness is not necessary for this argumentation.

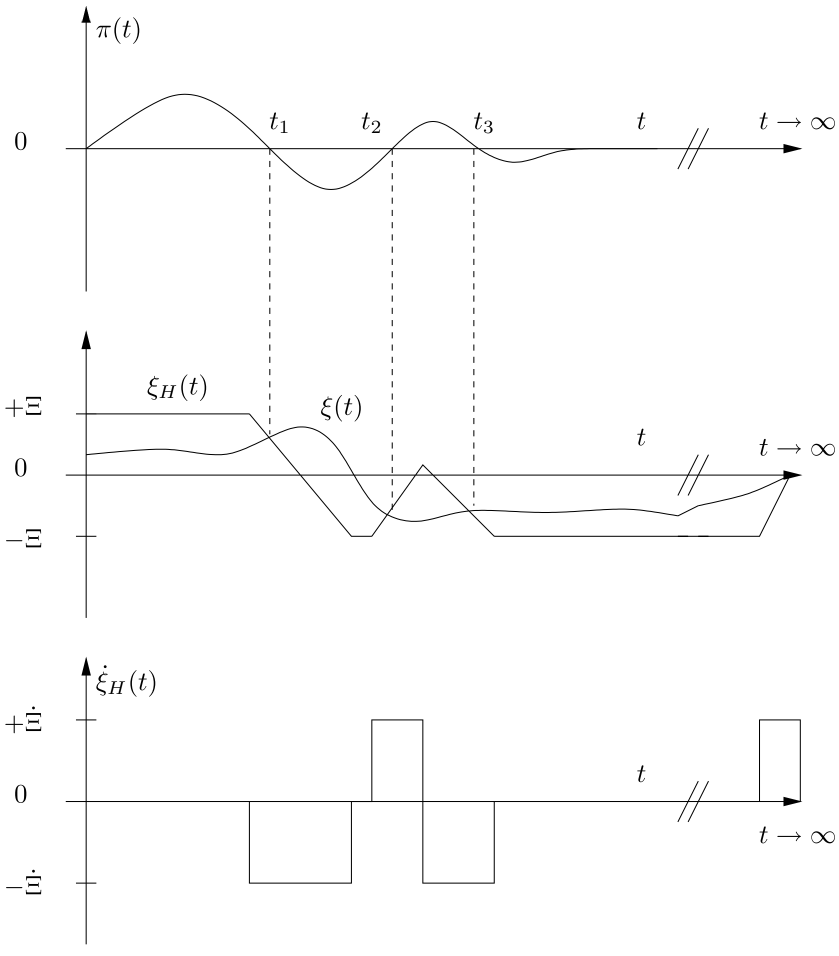

Figure 1: Construction of auxiliary input ξ_H for given input ξ.

Proof. For all $\xi \in \mathcal{A}(\Xi, \dot{\Xi})$ the following holds by construction of ξ_H, see Corollary 3.4 (3.):

$$\int_0^\infty \pi(\tau)\xi(\tau)d\tau \leq \int_0^\infty \pi(\tau)\xi_H(\tau)d\tau \tag{11}$$

and "=" holds only for $\xi \equiv \xi_H$ (except the case $\pi \equiv 0$). Assume $\xi_o \in \mathcal{A}(\Xi, \dot{\Xi}) \backslash \mathcal{A}_H(\Xi, \dot{\Xi})$, then the construction of an auxiliary input ξ_{oH} is possible (because ξ_o is admissible input). Applying eqn.(11) to $\xi = \xi_o$:

$$\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau < \int_0^\infty \pi(\tau)\xi_{oH}(\tau)d\tau \tag{12}$$

which contradicts the definition of Λ_m as the maximum output amplitude. Consequently $\xi_o \in \mathcal{A}_H(\Xi, \dot{\Xi})$. □

Until now, we did not construct a unique worst case input that leads to the maximum output amplitude, but we showed some necessary properties which are summarized in the following

3.6 Lemma The following necessary properties of the worst case input ξ_o hold:

1. The (derivative of the) worst case input has a pulse-shape: $\dot{\xi}_o(t) \in \{\pm\dot{\Xi}, 0\}$, and $\dot{\xi}_o(t) = 0$ implies $|\xi_o(t)| = \Xi$.
2. The width of the single pulses of $\dot{\xi}_o$ is constrained by $T = 2 \cdot \Xi / \dot{\Xi}$.
3. Two adjacent pulses have different signs.
4. $\lim_{t\to\infty} \xi_o(t) = 0$ and $|\lim_{t\to\infty} \dot{\xi}_o(t)| = \dot{\Xi}$.

The previous Lemma states mostly properties of $\dot{\xi}_o$. Partial integration in eqn.(10) gives an expression for the maximum output amplitude in this term, where s is the step response of the system (i.e. $\dot{s} = \pi$):

$$\Lambda_m = \int_0^\infty \pi(\tau)\xi_o(\tau)d\tau = \underbrace{\lim_{t\to\infty} \xi_o(t)s(t)}_{=0,\,\text{Lemma 3.6 (4.)}} - \underbrace{\xi_o(0)s(0)}_{=0,\,\text{eqn. (3)}} - \int_0^\infty s(\tau)\dot{\xi}_o(\tau)d\tau = -\int_0^\infty s(\tau)\dot{\xi}_o(\tau)d\tau \tag{13}$$

Despite the minus in the rhs of eqn.(13), $\Lambda_m \geq 0$ holds by Definition 3.2. It only appears due to partial integration.

Looking onto eqn.(13) and knowing the shape of the worst case input as stated in Lemma 3.6, the solution is quite intuitive: in order to make the integral maximal, put some pulses (of maximum width, see Corollary 3.4 (4.)) in the near of extrema of the step response: positive ones in the near of the minima and negative ones in the near of the maxima. In the following, we will state this formally. In order not to overload the discussion with technical details, we make the following

3.7 Temporary Assumption

1. Let the impulse response π have only a finite number N of zeros, i.e. the step response only a finite number of extrema.
2. Let the first extremum of s (i.e. the one with smallest argument t) be a (local) maximum.

Under these assumptions, the maximum amplitude Λ_m is given by:

$$\Lambda_m = \dot{\Xi} \sum_{i=1}^{N} (-1)^{i+1+k} \int_{t_i'}^{t_i''} s(t)dt + \Xi(-1)^{N+k} \lim_{t\to\infty} s(t). \tag{14}$$

The last part of the sum exists because the system is stable. The pairs (t_i', t_i'') refer to the unknown positions of the pulses of $\dot{\xi}_o$. Additionally, the sign of the pulses is still unknown, therefore we added $k \in \{0, 1\}$ in eqn.(14). Obviously the problem is solved, when exact location of the pulses and their sign are known. We make one more

3.8 Temporary Assumption Let the extrema of the step response have a distance $> 2T$, which ensures that all pulses have maximum width T, i.e. $t_i'' = t_i' + T$.

Assuming temporary assumptions 3.7 and 3.8, we are left with a maximum output amplitude Λ_m, only depending on the time stamps t_i' and the integer $k \in \{0, 1\}$: $\Lambda_m = \Lambda_m(t_i', k)$. The next theorem states necessary and sufficient conditions on the time stamps t_i' and the integer $k \in \{0, 1\}$, so that $\Lambda_m(t_i', k)$ is a maximum:

3.9 Theorem Necessary condition for $\Lambda_m = \Lambda_m(t_i', k)$ in eqn.(14) to be a maximum is $s(t_i') = s(t_i' + T)$, sufficient condition is $k = 0$.

Proof. Necessary for a maximum is $\frac{\partial \Lambda_m}{\partial t_i'} = 0$ for all i, which implies $s(t_i') = s(t_i' + T)$. Sufficient condition is that the Hessian matrix is negative definite, which leads to $\dot{\Xi}(-1)^{i+k}(\pi(t_i' + T) - \pi(t_i')) < 0$ and therefore $k = 0$, because of assuming a first local maximum of s in Assumption 3.7. Calculations are straightforward, only of technical nature and therefore omitted. □

Applying Theorem 3.9 to eqn.(14), we obtain the maximum output amplitude by

$$\Lambda_m = \dot{\Xi} \sum_{i=1}^{N} (-1)^{i+1} \int_{t_i'}^{t_i'+T} s(t)dt + \Xi(-1)^N \lim_{t \to \infty} s(t). \tag{15}$$

where we place the pulses $(t_i', t_i' + T)$, so that $s(t_i') = s(t_i' + T)$ holds with a negative sign under the maxima of s and with a positive sign under the minima of s.

3.10 Remark on temporary assumptions 3.7 and 3.8:

1. Suppose a first local minimum for s. In this case, the sufficient condition would lead to $k = 1$, i.e. to a change of signs for all pulses. We can treat these different cases by taking the absolute value of the sum in eqn.(15).

2. The stability of the system ensures $\pi(t) \to 0$ for large t, i.e. $s(t) = \text{const}$. In the case, that π has a infinite number of zeros the maximum output amplitude is given by eqn.(15) with an arbitrary precision when using arbitrary many zeros (i.e. $N \to \infty$). The proof is of technical nature and offers no deeper insight. We refer to [3, Appendix A2.1].

3. The "well-distinctness" of the extrema, assumed in Assumption 3.8 ensures that pulses with width T posed around the extrema of s will not intersect each other. Technically, this assumption simplifies the proof of Theorem 3.9. In general, we can prove that $\int_{t_i'}^{t_i''} \pi(t)dt = 0$ is necessary and sufficient for a maximum. Again, the proof is of technical nature and offers no deeper insight. We refer to [3, Appendix A2.2] for the details.

3.11 Remark For $\dot{\Xi} \to \infty$ (i.e. no restriction on the rate), the worst case input converges to the bang-bang input, as given in eqn. (5).

4 Numerical Solutions

Theorem 3.9 and Remark 3.10 give necessary and sufficient conditions to construct the worst case input. Therefore, one possible implementation is to construct the worst case input and then to calculated the maximum output amplitude using eqn. (15). Details of this approach are outlined in [3].

A more convenient way, however, is to approximate the solution by a linear optimization problem, which is stated in the following.

Suppose a grid of the time of all non-negative times, denoted as $\{t_k\}$ and evaluate the impulse response of the system Π at those time instances, denoted as $\{\pi_k\}$. Then the discrete time version of eqn. (10) is

$$\Lambda_m = \sum_{k=0}^{\infty} \pi_k \xi_{o,k}, \tag{16}$$

where $\xi_{o,k}$ is the input sequence, reversed in time. The task is to maximize eqn. (16) under the constraints (1,2), which can be approximated for the discrete time case by

$$\begin{aligned} -\Xi \leq \quad & \xi_{o,k} & \leq \Xi, \quad \forall k \geq 0, \quad (17) \\ -\dot{\Xi} \leq \quad & \frac{\xi_{o,k+1}-\xi_{o,k}}{t_{k+1}-t_k} & \leq \dot{\Xi}, \quad \forall k \geq 0. \quad (18) \end{aligned}$$

Obviously, the function (16) as well as the constraints (17,18) are *linear* in values of the input sequence evaluated on the time grid: $\xi_{o,k}$. Hence, maximization of (16) with respect to $\xi_{o,k}$ under the constraints(17,18) will deliver the optimal input sequence at times $\{t_k\}$. For practical reasons, the time grid $\{t_k\}$ can only cover a finite interval, say $[0, T_f]$, where T_f should be choosen larger than the largest time constant of the system Π. The Linear Program (16,17,18) over a finite time interval, however, yields the only the "last part" of the optimal input signal, as ξ_o is the *time reversed* input signal. The first part of the optimal input sequence can be constructed as in Lemma 3.6 (4.), or, in the case that we are only interested in the maximum output, rather than the optimal input sequence, by calculating the second part of the sum in eqn. (15).

5 Multivariable Case

We extend the previous result to the case of multivariable systems, i.e. ξ and λ are now vector valued signals. What we have in mind is the treatment of multivariable control systems with constraint control signals, i.e. we regard the control signal as output, $\lambda = u$, the (external) the reference signal as input, $\xi = r$, and $\Pi(s)$ is the transfer function defined by $u = \Pi \cdot r = K(I + GK)^{-1} \cdot r$, assuming the standard control control system with negative feedback, controller K and plant G. Therefore, it is useful to restrict the input ξ componentwise, in order to handle each reference channel separately from the others:

$$\begin{aligned} |\xi(t)| \quad & \preceq \quad \Xi, \; t > 0 \quad (19) \\ |\dot{\xi}(t)| \quad & \preceq \quad \dot{\Xi}, \; t > 0 \quad (20) \\ \xi(t) \quad & = \quad 0, \; t \leq 0. \quad (21) \end{aligned}$$

in complete analogy to eqns.(1-3). Read $\preceq$ as a componentwise $\leq$ and evaluate $|\cdot|$ in this context componentwisely. Consequently, we call the set of all signals ξ fulfilling these constraints $(\Xi, \dot{\Xi})$-admissible, with $\Xi, \dot{\Xi}$ are now being vectors with positive entries.

Furthermore, we define the maximum amplitude of the n-dimensional output $\lambda = (\lambda_1, \dots, \lambda_n)^T$ componentwisely as

$$\Lambda_m := (\Lambda_{1,m}, \dots, \Lambda_{n,m})^T \quad (22)$$

where $\Lambda_{i,m}$ is defined as in Lemma 3.1 (the input ξ in eqn. (4) now being a vector valued signal). This componentwise definition of the maximum output amplitude in the multivariable case is strictly motivated by the design of multivariable controllers: in this case $\lambda = u$ is the control signal and eqn. (22) enables us to consider different bounds on different control channels, in contrast to a possible definition using for instance 1-norm or ∞-norm.

The remaining question is, how the results gained in sec. 3 and 4 can be used in the multivariable setup. Therefore, we first look onto a system with one output λ and k inputs $\xi = (\xi_1, \dots, \xi_k)^T \in \mathcal{A}(\Xi, \dot{\Xi})$. Then $\lambda(s)$ is given by

$$\lambda(s) = \Pi_1(s) \cdot \xi_1(s) + \dots + \Pi_k(s) \cdot \xi_k(s). \quad (23)$$

We abbreviate the response to each of the input channels by $\tilde{\lambda}_i(s) = \Pi_i(s) \cdot \xi_i(s)$. Now we are looking for the maximum output amplitude Λ_m. Using eqn. (23), the maximum output amplitude is given by

$$\Lambda_m = \sum_{i=1}^{k} \tilde{\Lambda}_{i,m}. \quad (24)$$

It follows directly, that Λ_m is achieved for a certain *vector* $\xi = (\xi_1, \dots, \xi_k)^T \in \mathcal{A}(\Xi, \dot{\Xi})$, as all input channels can be chosen independently to maximize their contributions $\tilde{\lambda}_i$ in eqn. (24).

In the multivariable case with n outputs, we simply apply the first step for each component: according to the definition, the components $\Lambda_{i,m}$ of Λ_m can be calculated as in equation (24). We should, however, note that when using this approach, the maximum output amplitude will not be reached in all channels in one "operation mode". Consider for instance a SIMO system, then the maximum output amplitude of channels i, j may be achieved when feeding the system with certain admissible input signals ξ^i, ξ^j, which are in general different form each other (but still both admissible!). Thus, when feeding the system with input signal ξ^i, output channel i will achieve its maximum amplitude, but $\sup_t \|\lambda^j\| = \sup_t \|\pi^j \star \xi^i\| < \sup_t \|\pi^j \star \xi^j\|$. This "overestimation" appears infact because the definition of the maximum amplitude in eqn. (22) employs *no norm*.

6 Illustrative Example

We examine the system represented by the transfer function

$$\Pi(s) = \frac{s^2 + 0.4}{s^2 + 1.4s + 1.0},$$

with input constraints $\Xi = 1.0$ resp. $\dot{\Xi} = 0.8$. According to Lemma 3.6 (2.), the maximum pulse width is $T = 2 \cdot \Xi / \dot{\Xi} = 2.5$. Numerical solution by construction of the worst case input yields the maximum output amplitude of

$$\Lambda_m = 0.76. \tag{25}$$

Fig. 2(b) shows the construction of the worst case input with pulses of $\dot{\xi}_o$ located at $[0, 1.20]$, $[1.20, 2.39]$, $[5.24, 7.74]$ and of width $T/2$ at infinity (in reversed time) with signs "- + - +" following the max-min-max sequence of the step response, cf. fig. 2(a). For simulation of this worst case input we need to reverse this reversed time, therefore we choose the "infinite" time to $t_\infty = 120$. With this transformation, we obtain the worst case output depicted in fig. 2(c) with a maximum amplitude of $\lambda(t_m) = 0.73$ for $t_m = 108.80$ as a good approximation for the maximum amplitude as calculated in eqn.(25).

7 Conclusions and Related Works

We gave necessary and sufficient conditions of the worst case input with bounded amplitude and rate, that produces the maximum output amplitude for a given (stable) multivariable system. A numerical algorithm was formulated to construct this worst case input and to calculate the maximum output amplitude. This is a necessary and important step within several non-conservative controller design procedures for systems with hard bounds on the control signal, as we are now able to calculate the maximum control signal and adapt the controller in such a way, that we meet the prescribed bound on the control signal. Moreover it enables us to check the maximum amplitude of an arbitrary signal within the control system for an already existing controller. These control applications are presented in detail in [3, 4, 6, 5].

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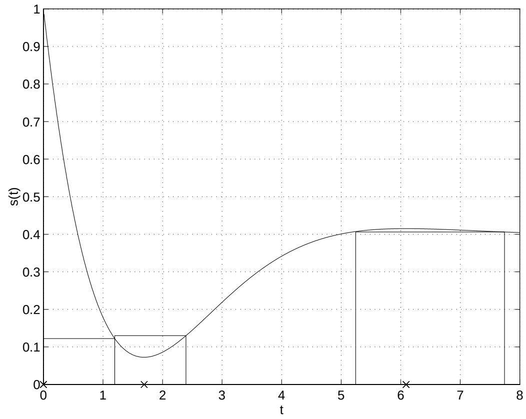
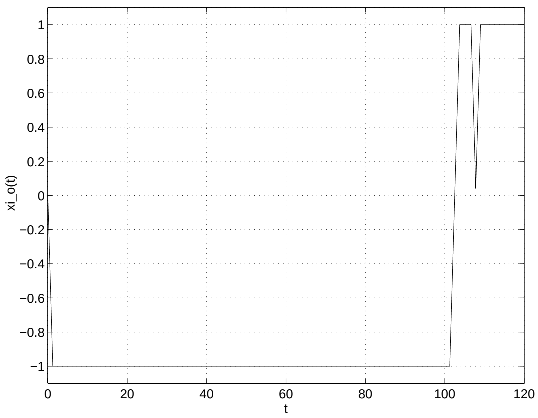
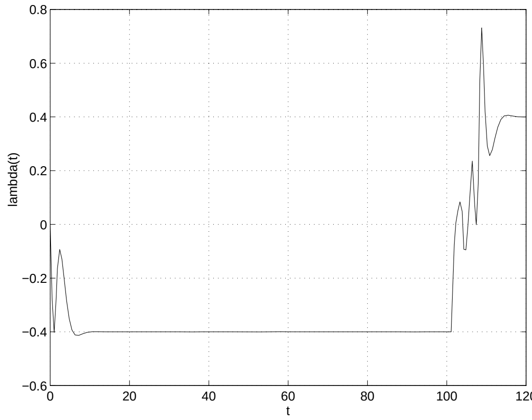


(a) Step response s with calculated position of the pulses for the worst case input.



(b) Worst case input.



(c) Worst case output.

Figure 2: Illustrative example: construction of the worst case input and output.

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