Control of Multivariable Systems with Hard Constraints

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Abstract

A general framework for the design of multivariable control systems subject to hard constraints on each control channel is developed. The design procedure is an extension of the well-known $\mathcal{H}_\infty$ Loop Shaping Design Procedure and is based on the calculation of the maximum possible control amplitude for a certain class of reference signals. Special attention is given to the adaption of the design weights in order to meet the prescribed bounds on the control signal. A multivariable simulation example, the control of the vertical dynamics of an aircraft, illustrates the suggested procedure.

1 Introduction and Motivation

Most practical control problems are dominated by *hard bounds*. Valves can only be operated between fully open and fully closed, pumps and compressors have a finite throughput capacity and tanks can only hold a certain volume. These input- or actuator-bounds convert the linear model into a nonlinear one. Exceeding these prescribed bounds causes unexpected behavior of the system – large overshoots, low performance or (in the worst case) instability.

Design of controllers for systems with hard constraints is a quite vivid area of research, see for example the recent books [20, 23] or the overview paper [2] and the references therein. The problem has been addressed within the Model Predictive Control community [10] and another popular approach is to use so-called Anti Windup Bumpless Transfer schemes [1, 11].

Controller designs that encounter the saturation effect a-priori are usually separated into two categories: (1) designs that prevent saturation of the control signal and therefore enjoy a linear framework (as long as plant and controller are linear) and (2) methods that allow saturation and are therefore facing a *nonlinear* setup. In the second case, analysis (in terms of stability, controllability and feasibility) of this nonlinear system is discussed in [6, 21, 22]. Design schemes that handle saturations using a (nonlinear) control law have recently been proposed in [19, 3] for instance.

This work clearly employs the first – saturation avoiding – philosophy. To solve the constraint control problem, one implicitly has to restrict the amplitude of all external signals – independent of the technique used in particular. Our approach, however, differs from the ones cited above by imposing an additional restriction on the rate of the external signals. In many practical situations, this is a more accurate description (than without rate restriction) of all external signals, possibly arising during runtime: in the example of the tank from above, not only the liquid-level is bounded (by the tanks height), additionally the liquid cannot change its level arbitrarily fast. Therefore, a design, directly based on this description will avoid a conservative control system. Using the same description of external signals, somewhat related works studied energy bounds (so-called soft bounds) instead of hard bounds [8] or allow some process noise [13]. Optimal single-input single-output control systems with respect to hard constraints have been studied in [17]. Robust and constrained single-input single-output systems have been studied in [14, 16].

This paper is organized as follows: Sec. 2 discusses the calculation of the maximum possible amplitude of the control signal in a MIMO system. Then, Sec. 3 states the well-known $\mathcal{H}_\infty$ Loop Shaping Design Procedure, extended for the design of systems with bounded control signals. The crucial point within the Loop Shaping Design Procedure is the systematic adjustment of the design weights. In our case the task becomes tougher, when the prescribed hard bounds (on the control signal) are not met. We present a general guideline in Sec. 4. The usage of this guideline is illustrated for a multivariable simulation example in Sec. 5: the control of the vertical dynamics of an aircraft.

2 Multivariable Systems and their Maximum Control Signal

As motivated in the introduction, we study control systems with reference signals, bounded in amplitude and speed. Aim of the controller design is to handle hard bounds of the control signal. The signals are depicted in the standard control loop in Fig. 1. We give the following definitions, which are straightforward extensions of those in[13] to the multivariable case:

2.1 Definition (Admissible Reference Signal) Given $0 \preceq R, \dot{R} \in \mathbb{R}^n$. Then a vector-valued reference signal r is called $(R, \dot{R})$-admissible, when the following properties hold:

1. $|r(t)| \preceq R$ for all $t > 0$ and
2. $|\dot{r}(t)| \preceq \dot{R}$ for all $t > 0$,

whereas $\preceq$ denotes componentwise $\leq$ and $|\cdot|$ has to be evaluated componentwisely in this context. The set of all $(R, \dot{R})$-admissible reference signals is denoted by $\mathcal{A}(R, \dot{R})$.

2.2 Definition (Maximum Control Signal) Given the internally stable standard control loop as in Fig. 1. We call

$$u_{\max} := \begin{pmatrix} \sup\{||u_1||_\infty;\ \forall r \in \mathcal{A}(R, \dot{R})\} \\ \vdots \\ \sup\{||u_n||_\infty;\ \forall r \in \mathcal{A}(R, \dot{R})\} \end{pmatrix} \qquad (1)$$

the Maximum Control Signal.

The definition of the admissible reference signal is quite straightforward from the motivation. The componentwise definition of the maximum control signal enables us to handle hard bounds for each of the control signals, which is a clear advantage compared to the ∞-*norm* of a vector-valued signal, for example.

The core of our design procedure is that we are able to calculate the Maximum Control Signal in a given multivariable control system *exactly*. To show this, we state the following result from [13], which holds for the SISO case:

2.3 Theorem and Algorithm (Calculation of the Maximum Control Signal) Given a linear and time invariant SISO control system with reference signal $r(t)$ and control signal $u(t)$. Let the reference signal be $(R, \dot{R})$-admissible and denote the transfer function from reference to control signal with H. Then:

(a) There exists an algorithm that determines the maximum control signal $u_{\max}$ according to Definition 2.2 of this system for *all* admissible reference signals.

(b) An $(R, \dot{R})$-admissible input exists, so that $u_{\max}$ is achieved.

The algorithm outlined in the original work [13] constructs a worst case input r. A throughout treatment, along with an alternative numerical solution of the problem, is given in [15]. Independent of the numerical solution, Algorithm 2.3 can be used to determine the maximum control signal $u_{\max}$ in a SISO system, when the reference signal $r(t)$ is $(R, \dot{R})$-admissible.

The remaining question is, how the results gained in Theorem 2.3 can be used in the multivariable setup (cf. also the discussion in [15]). Therefore, we first look onto a system with one control signal u and k reference inputs $r = (r_1, \ldots, r_k)^T \in \mathcal{A}(R, \dot{R})$. Then $u(s)$ is given by

$$u(s) = H_1(s) \cdot r_1(s) + \cdots + H_k(s) \cdot r_k(s). \qquad (2)$$

We abbreviate the response to each of the input channels by $\tilde{u}_i(s) = H_i(s) \cdot r_i(s)$. Now we are looking for the maximum control signal $u_{\max}$. Using eqn. (2), this maximum is given by

$$u_{\max} = \sum_{i=1}^{k} \tilde{u}_{i,\max\cdot}, \qquad (3)$$

where the $\tilde{u}_{i,\max}$ can be calculated using Theorem 2.3(a), as H_i is single-input single-output. It follows directly from Theorem 2.3(b), that $u_{\max}$ is achieved for a certain *vector* $r = (r_1, \ldots, r_k)^T \in \mathcal{A}(R, \dot{R})$, as all input channels can be chosen independently to maximize their contributions $\tilde{u}_i$ in eqn. (3). In the multivariable case with n control signals, we simply apply the first step for each component.

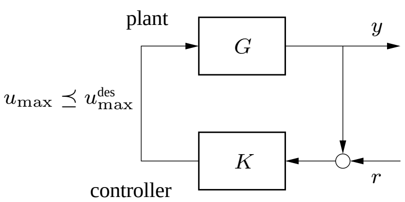

Figure 1: Multivariable control system with constraint control signal u.

3 $\mathcal{H}_\infty$ Loop Shaping for Multivariable Systems with Constraints

The result presented above in Sec. 2 enables us to calculate the maximum control signal of a control system, when the external signal, i.e. the reference signal fulfills these constraints. We will exploit this to extend the well known $\mathcal{H}_\infty$ Loop Shaping Design Procedure by McFarlane & Glover [12], as depicted in Fig. 2, to the case of constraint control signals:

3.1 Extended Loop Shaping Design Procedure Given a (multivariable) plant G, restrictions $R, \dot{R} \succeq 0$ for the reference signal and a desired maximum control signal $u_{\max}^{\text{des}} \succ 0$ for all control channels, we will perform the Loop Shaping Design Procedure (LSDP) in the following way:

1. Choose a performance factor f and weights W_1 and W_2 to shape the plant.
2. Controller-design for the shaped plant and calculation of the stability margin ϵ.
3. Calculation of the final controller (including the weights).
4. Decide whether the design-objectives are fulfilled or not:
   - Is the stability margin ϵ large enough?
   - Are the performance-objectives fulfilled?
   - Does $u_{\max} \preceq u_{\max}^{\text{des}}$ hold?

   If not, choose other weights W_1, W_2 (and/or another performance factor f) and go back to the first step.

Theorem 2.3 enables us to determine the maximum control signal $u_{\max}$ within the control loop, which allows the check for the desired bound on the control signal in step 4. This *additional* check, a-posteriori of nature, is the basic difference to the classical LSDP. We refer to this procedure in the following with Extended Loop Shaping Design Procedure (ELSDP).

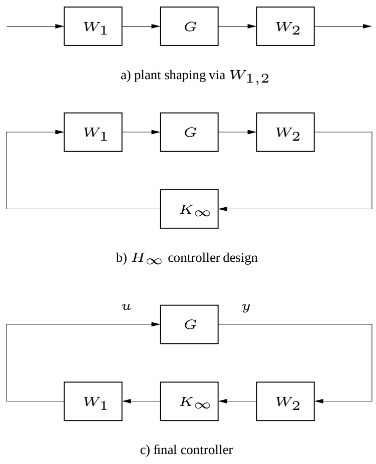

Figure 2: $\mathcal{H}_\infty$ Loop shaping and controller design in three steps.

4 Adjustment of the Design Weights

We discuss a question neglected in the last section: after a first choice of the design weights, our design objectives are usually *not* fulfilled – the weights have to be adjusted. In the case of a too small stability margin ϵ the strategy is clear from the classical LSDP: because there is no explicit relation known between achieved stability margin and weights, we have to examine the singular values of the shaped plant and the achieved open loop. In the frequency range with a significant difference, our weights are incompatible with the plant and have to be adjusted. Detailed studies on this topic for the "classical" LSPD are described for example in [4, 5, 9].

In the following we discuss the remaining open question: is there a strategy for correct and systematic adjustment of the weights, when the maximum control signal is still too large after a loop shaping step?

Within loop shaping, we work on the singular values of different interesting transfer functions. Thus we are searching for a relation between the singular values of the transfer function and the maximum control signal $u_{\max}$. The relation between reference signal and control signal is given by $u(s) = H(s) \cdot r(s)$ and $u(t) = h(t) \star r(t)$ respectively, where "$\star$" denotes the convolution and a state space representation is given by $H = (A, B, C, D)$. Because of the componentwise definition of the maximum control variable, we restrict our following examinations to the case of a single control variable. The generalization follows immediately by componentwise usage (cf. Definition 2.2). In [14], we showed the following useful relation for the case of a stable and strictly proper transfer function H:

$$||u||_\infty \leq 2n \cdot ||H||_\infty \cdot ||r||_\infty,$$

where n denotes the McMillan degree of H. In [16], we extended this to the case of proper transfer functions:

4.1 Theorem [16] In the case of a stable and proper transfer function H with input r and output u,

$$||u||_\infty \leq (2||H||_\infty + 3d) \cdot n \cdot ||r||_\infty \qquad (4)$$

holds with $D = [d_1, ..., d_n]^T$ and $d := \max_i |d_i|$.

Proof. See [16]. The extension to multivariable systems is quite straightforward and omitted.

We now turn back to our final aim: the relation between the singular values of H and $||u||_\infty$. As the control loop is internally stable, the transfer function H is stable. Following equation (4), we see that decreasing the ∞-norm of H decreases an upper bound for the maximum control signal.

Suppose, the maximum control signal is too high after a loop shaping step. We then have to decrease the maximum singular value of H in the frequency range where the ∞-norm appears. In the case of a too low maximum control signal, we have to increase the maximum singular value in that frequency range. We point out, that this affects only an *upper bound* for the maximum control signal. In general, there might be much space between the both sides of eqn.(4). We only use it as a guideline for the adjustment of the weights in the "correct direction" and in the correct frequency range. The practical value of this guideline is shown in the example. However, the initial choice of the weights should attack the general shape of the open loop and is discussed in context with original Loop Shaping Design Procedure by McFarlane & Glover [12] and related works [4, 5, 9] or textbooks on robust control [7].

4.2 Remark It seems very straightforward to lower the gain of a transfer function in order to lower the "size" of the output signal. Note, however, that the norm, induced by the (operator-) $\mathcal{H}_\infty$ norm is the 2-norm (in the signal space), in which we are not interested in the first place. The explicit relation between $\mathcal{H}_\infty$ norm of the transfer function and ∞- or $\max$-norm of the signals is given in equation (4) and, to our best knowledge, not available elsewhere.

4.3 Remark Equation (4) is still useful for the proper adaption of design weights, even when we leave the framework we are using in this work. Describing an upper bound, it is *independent* of the restrictions on the reference signal or the exact computability of the maximum control signal.

5 Illustrative Example: Vertical Dynamics Control of an Aircraft

We study a multivariable continuous time plant, an aircraft model, examined in great detail in [12]. The plant has three inputs and outputs, two complex conjugate pole pairs and a pole in the origin. Throughout the example, we determine suboptimal controllers ($f = 1.1$). McFarlane & Glover demand, additionally to performance and stability objectives, the following componentwise bounds on the control signal u:

$$|u_1(t)| < 40, |u_2(t)| < 10, |u_3(t)| < 40, \forall t \geq 0. \qquad (5)$$

To solve this problem with the proposed extension of the Loop Shaping procedure, we restrict the reference signal by the following values:

$$R = [1, 1, 1]^T, \quad \dot{R} = [5, 11, 3]^T. \tag{6}$$

5.1 Analysis of the McFarlane & Glover Design

We start up with the design [12, Sec. 7.4.3, design 2] based on the following diagonal weight $W(s) = \text{diag}\{w_1(s), w_2(s), w_3(s)\}$ with:

$$w_1(s) = w_3(s) = 24w_c, w_2(s) = 12w_c, w_c(s) = \frac{s + 0.4}{s}$$

at the plant output. The usage of *one diagonal weight* ensures a better oversight during Loop Shaping. Therefore, we restrict ourselves to this class of weights. The gain increases the open loop and thus increases the $0dB$ crossover frequency. The integral action will improve the low frequency performance. Looking at the singular values of the unshaped plant (see Fig. 3, upper plot), the zeros at -0.4 limit the integrator to the low frequency range, so that a too high roll-off rate near the crossover frequency is prevented. This would cause poor robustness properties (i.e. a small stability margin) or even instability (known from Bode's Gain-Phase relations). The weight W leads to a eleventh order controller K and the resulting system has the following maximum control amplitude:

$$u_{\max} = [26.53, 10.49, 61.42]^T$$

for all admissible reference signals obeying (6). The stability margin is $\epsilon = 0.38$ and Fig. 3 shows singular values[1] of some closed loop functions. We observe from the above computed value $u_{\max}$, compared to (5), that we can "effort" much more in the first control channel, while the amplitude in the second one is slightly, and in the third one is quite too large. Aim of a proper weight adaption will be, according to the recommendations in Sec. 4, to increase the gain of the transfer function from (all) reference signals to the first control channel, and to decrease gain of the transfer functions from (all) reference signals to the second and third control channel respectively. These gains are reported in Fig. 4.

5.2 Adjustment of the Weights

Now, we adjust the weight W in order to achieve the meet maximum control amplitude. According to the results derived in Sec. 4, we are looking at the singular values in Fig. 4. We see, that they achieve their maximum at frequencies around $10rad/s$. As motivated above, we have to increase the first and to decrease the other ones by proper adaption of the weight W. Hence, we increase the first diagonal entry w_1 and decrease the other two entries of weight W in this frequency range respectively. The amplitude of the adapted weight is reported in

[1]Throughout this example, we only show largest and smallest singular values.

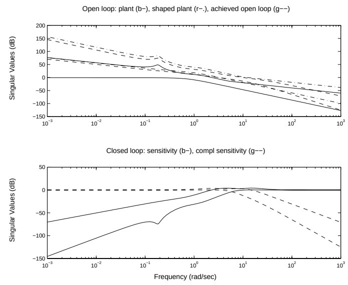

Figure 3: McFarlane & Glover design: Upper plot: singular values of plant (solid), shaped plant (dash dotted) and achieved open loop (dashed). Lower plot: singular values of sensitivity (solid) and complementary sensitivity (dashed).

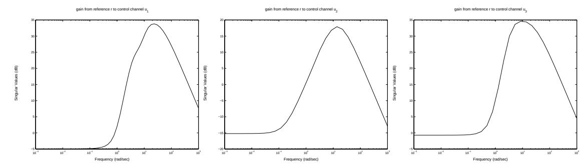

Figure 4: McFarlane & Glover design: gains of the transfer functions from reference signal (vector) to control channels 1-3 *separately* (left to right).

Fig. 5. Using the adapted weight W^a, we obtain a maximum control amplitude of:

$$u_{\max} = [39.06, 10.00, 39.62]^T$$

and achieve a stability margin of $\epsilon^a = 0.30$. The resulting controller K^a is of order 35 and can easily obtained using MATLABs μ Toolbox command `ncfsyn`.

Our design objectives regarding the constraint control variable are fulfilled! Fig. 6 shows the singular values of the control system made up with controller K^a. The singular values of the transfer functions to the single components of the control signal, have been adapted correctly in the frequency range in question, as reported in Fig. 7: increased from $34dB$ to $37dB$ for the first control channel and decreased from $17.5dB$ to $17dB$ ($35dB$ to $27dB$) in the second (third) control channel.

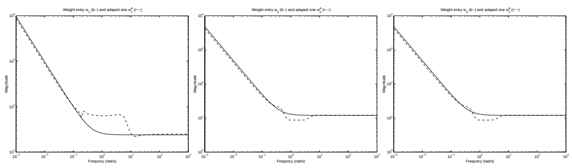

Figure 5: Three diagonal entries (left to right) of the McFarlane and Glover weight W (solid) and the adapted weight W^a (dashed): The first entry was increased while the other ones have been decreased in the frequency range of interest.

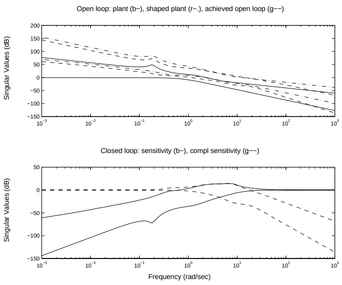

Figure 6: Design with adapted weights: Upper plot: singular values of plant (solid), shaped plant (dash dotted) and achieved open loop (dashed). Lower plot: singular values of sensitivity (solid) and complementary sensitivity (dashed).

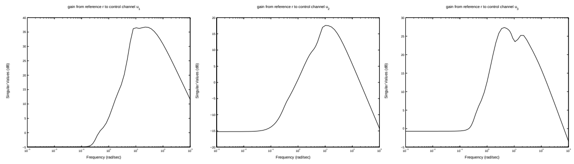

Figure 7: Adjusted design: gains of the transfer functions from reference signal (vector) to control channels 1-3 *separately* (left to right).

5.3 Simulation Studies

We simulate both closed loop systems (employing controllers K and K^a respectively), including a saturation nonlinearity, and assuming the presence of a white band limited noise with power 10^{-6} at the plant output (cf. Fig. 1). We stimulate both systems with an $\left([1,1,1]^T, [5,11,3]^T\right)$-admissible reference signal. The reference/output signals and control signals are reported in Figs. 8 and 9 respectively. We observe, that the control system including controller K runs into saturation two times (in the third control channel), while the control system made up with controller K^a does not. This fulfills our expectation, as we reported a much too large third control amplitude in the earlier analysis.

We observe no significant difference when tracking output channels 2 and 3. The tracking of channel 1, however, has improved when applying the adjusted controller (or, in turn this means that running into saturation degrades the performance of the system with the original controller being involved). Similar behaviour can be observed in other simulation studies.

6 Conclusions

We studied the control of multivariable control systems with hard bounded control signals. One main point within the extension of the $\mathcal{H}_\infty$ Loop Shaping was the calculation of the maximum control signal for the set of admissible reference signals. The other main point was the systematic adaption on the weights with respect to the control signals bound by deriving an explicit relation between design weight and maximum control signal. The presented framework extends previous works to the

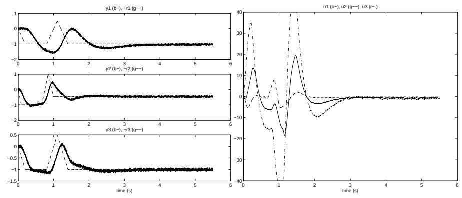

Figure 8: Simulation of control system with controller K. Left: reference (dashed) and output signals (solid). Right: control signals 1 (solid), 2 (dashed) and 3 (dash-dotted).

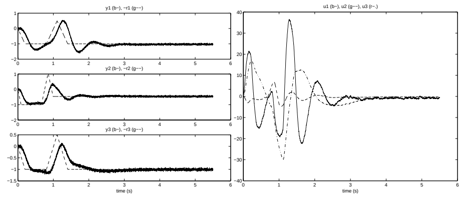

Figure 9: Simulation of control system with adapted controller K^a. Left: reference (dashed) and output signals (solid). Right: control signals 1 (solid), 2 (dashed) and 3 (dash-dotted).

multivariable case with hard bounds on each control channel.

An interesting question is how to guarantee the control signal within the suggested bounds for the *set* of uncertain plant - in this paper, it is only guaranteed for the nominal plant. However, the simulation studies in these works showed sufficient behavior of the control signal even in the case of a parameter variation. In [18], a method for *a-priori* incorporation of uncertain plants is proposed.

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